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Complex Representations of GL(2, K) for Finite Fields K

Ilya Piatetski-Shapiro

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FOREWORD

These are Lecture Notes of a course that I gave in Tel-Aviv University. The aim of these Notes is to present the theory of representations of GL(2,K) where K is a finite field. However, the presentation of the material has in mind the theory of infinite dimensional representations of GL(2,K) for local fields K.

I am very grateful to Moshe Jarden who took these Notes and worked them out. Without him it would have been completely impossible to prepare them.

This course and its Notes are the first outcome of the Cissie & Aaron Beare Chair in Algebra and Number Theory.

Ilya Piatetski-Shapiro
Tel-Aviv
November, 1982

Contents

Introduction

The aim of these notes is to give a description of the complex irreducible representations of the group $G = GL(2,K)$, where K is a finite field with $q > 2$ elements. In addition these notes should also serve as a motive for the study of the representation of $GL(2,K)$, where K is a local field. Therefore an attempt has been made to reprove theorems by not explicitly using the finiteness of K.

A central role in the description of the representations of G is played by the Borel subgroup consisting of all the matrices

$$b = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \qquad \alpha,\delta \in K^{\times}, \qquad \beta \in K .$$

If μ_1,μ_2 are characters of $K^{\times}$, then a character μ of B can be defined by $\mu(b) = \mu_1(\alpha)\mu_2(\delta)$. Let $\hat{\mu} = \mathrm{Ind}_B^G\mu$ be the induced representation. If $\mu_1 = \mu_2$, then $\hat{\mu}$ splits as the direct sum of a one-dimensional representation $\rho'_{(\mu_1,\mu_1)}$ which is given by formula $\rho'_{(\mu_1,\mu_1)}(g) = \mu_1(\det g)$, and a q-dimensional irreducible representation $\rho_{(\mu_1,\mu_1)}$. There are $q-1$ representations of each kind. If $\mu_1 \neq \mu_2$, then $\hat{\mu} = \rho_{(\mu_1,\mu_2)}$ is an irreducible representation of dimension $q+1$. There are $\frac{1}{2}(q-1)(q-2)$ representations of this kind. Irreducible representations that are not of the above types are of dimension $q-1$ and are called cuspidal representations. They are however also connected with linear characters in the following way. Let L be the unique quadratic extension of K and let ν be a character of $L^{\times}$ for which there does not exist a character χ of $K^{\times}$ such that $\chi(N_{L/K}z) = \nu(z)$ for every $z \in L^{\times}$. Such a ν is said to be non-decomposable. For each non-decomposable character ν of $L^\times$ we explicitly construct an irreducible representation ρ_ν of G and prove that it is cuspidal. Conversely, we prove that every cuspidal representation of G is of the form ρ_ν for some non-decomposable character ν of $L^\times$. Thus there are $\frac{1}{2}(q^2-q)$ cuspidal representations.

The connection between the irreducible representations of G and the characters of $K^\times$ and $L^\times$ gives rise to a reciprocity law. Let $W(L/K) = L^\times \cdot G(L/K)$ be the semi-direct product of $L^\times$ by $G(L/K)$. The irreducible representations of $W(L/K)$ (which is called the small Weil group) of dimension ≤ 2. The announced reciprocity law is a natural bijection between the two-dimensional representations of $W(L/K)$ (including the reducible ones) and the irreducible representations of G of dimension > 1.

Next we attempt to give explicit models for the irreducible representations of G. Let ψ be a non-unit character of K^+. The additive group K^+ can be canonically identified with the subgroup U of G consisting of all the matrices of the form

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \qquad \beta \in K .$$

Therefore ψ can be also considered as a character of U. We prove that $\mathrm{Ind}_U^G \psi$ splits into the direct sum of all irreducible representations ρ of G of dimension > 1; each ρ appears with multiplicity 1. The space V_ρ on which ρ acts can therefore be embedded into $\mathrm{Ind}_U^G V_\psi$. Thus to each $v \in V_\rho$ there corresponds a function $W_v\colon G \to \mathbb{C}$ such that $W_v(ug) = \psi(u)W_v'(g)$ for every $u \in U$ and $g \in G$. The action of ρ on these functions is given by $W_{\rho(s)v}(g) = W_v(gs)$. The collection of all the W_v is called a Whittaker model for ρ. It has the following property: For all characters ω of $K^\times$ except possibly two there exists complex numbers $\Gamma_\rho(\omega)$ such that

$$(1) \qquad \Gamma_\rho(\omega) \sum_{x \in K^\times} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \omega(x) = \sum_{x \in K^\times} W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x)$$

for every $v \in V_\rho$. If ρ is a cuspidal representation, then $\Gamma_\rho(\omega)$ is defined for every ω.

Among the Whittaker functions for ρ there is a special one, J_ρ, called the Bessel function of ρ, that satisfies

$$J_\rho(gu) = J_\rho(ug) = \psi(u)J_\rho(g) \quad \text{for } u \in U, \; g \in G .$$

Further, $J_\rho(1) = 1$ and $J_\rho(u) = 0$ for $u \in U$ and $u \neq 1$. Substituting this function for W_v in (1) we have

$$\Gamma_\rho(\omega) = \sum_{x \in K^\times} J_P \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \omega(x) .$$

This formula is then used in order to express $\Gamma_\rho(\omega)$ in terms of Gauss sums: If $\rho = \rho_{(\mu_1,\mu_2)}$ is a non-cuspidal representation of G, then

$$\Gamma_\rho(\omega) = \frac{\omega(-1)}{q} G_K(\mu_1^{-1}\omega^{-1},\psi) G_K(\mu_2^{-1}\omega^{-1},\psi) .$$

If $\rho = \rho_\nu$ is a cuspidal representation, then

$$\Gamma_\rho(\omega) = \frac{\nu(-1)}{q} G_L(\nu\cdot(\omega\circ N_{L/K})^{-1}, \; \psi\circ Tr_{L/K}) .$$

The Gauss sum $G_K(\chi,\psi)$ is defined for a character ψ of $K^\times$ and a character ψ of K^+ by

$$G(\chi,\psi) = \sum_{x \in K^\times} \chi(x)\psi(x) .$$

In particular it follows that in every case $|\Gamma_\rho(\omega)| = 1$.

All these results are finally applied in order to compute the characters table for G.

Chapter 1. Preliminaries: Representation theory; the general linear group

In the first three sections of this chapter we bring all the definitions and theorems about linear representations of finite groups that we need in these notes. We refer to Serre [2] and to Lang [1] for the proofs. The remaining two sections are devoted to a description of the group-theoretical properties of $GL(2,K)$, where K is a finite field.

1. Linear representations of finite groups.

Let V be a finite dimensional vector space over the field $\mathbb{C}$ of the complex numbers. Denote by $\mathrm{Aut}(V)$ the group of all automorphisms of V. Let G be a finite group. A linear representation of G in V is a homomorphism ρ of G into $\mathrm{Aut}(V)$. V is said to be the representation space of ρ and is also denoted by V_ρ. We shall also say that G acts on V_ρ through ρ. The dimension of ρ is defined to be the dimension of V_ρ and is denoted by $\dim \rho$. Two representations ρ and ρ' of G are said to be isomorphic, if there exists an isomorphism $\theta: V_\rho \to V_{\rho'}$ such that $\theta \circ \rho(g) = \rho'(g) \circ \theta$ for every $g \in G$. We shall usually identify isomorphic representations.

A representation of G of dimension 1 is a homomorphism μ of G into the multiplicative group $\mathbb{C}^\times$ of $\mathbb{C}$. Such a representation is called in these notes a character of G. In particular, the unit character is the homomorphism of G into $\mathbb{C}^\times$ obtaining the value 1 for every $g \in G$.

Let ρ be a representation of G and let H be a subgroup of G. Suppose that μ is a character of H for which there exists a non-zero $v \in V_\rho$ such that $\rho(h)v = \mu(h)v$ for every $h \in H$. Then μ is said to be an eigenvalue of H (with respect to ρ) and v is said to be an eigenvector of H that belongs to μ.

Again consider a representation ρ of G and let V' be a subspace of $V = V_\rho$ which is left invariant by $\rho(g)$ for every $g \in G$. In this case we say that V' is left invariant by G or that V' is a G-subspace of V. Then the restriction map of $\rho(g)$ to V' gives rise to a representation ρ' of G with V' as its representation space. This representation is said to be a sub-representation of ρ and we write $\rho' \leq \rho$. By a theorem of Maschke V' has a complement in V, i.e., there exists another G-subspace V'' of V such that $V = V' \oplus V''$ (c.f. Serre [2, p. 18]). Let ρ'' be the corresponding subrepresentation of ρ. Then ρ is said to be a direct sum of ρ' and ρ'' and we write $\rho = \rho' \oplus \rho''$. Clearly $\dim \rho = \dim \rho' + \dim \rho''$. The direct sum of n representations of G, all isomorphic to ρ, is denoted by $n\rho$. A representation ρ of V is said to be irreducible if it does not have a sub-representation ρ' of a lower dimension. By the theorem of Mascke this is equivalent to saying that ρ cannot be decomposed as a direct sum $\rho = \rho' \oplus \rho''$ with $\dim \rho' < \dim \rho$. It follows that every representation ρ of G can be represented as a direct sum $\rho = \bigoplus_{i=1}^{k} n_i \rho_i$, where the ρ_i are distinct (i.e., non-isomorphic) irreducible representations of G. This decomposition of ρ is unique, up to the order of the summands (c.f., Serre [3, p. 34]).

There are only finitely many irreducible representations $\rho_1, \ldots, \rho_n$ of G. Their number h is equal to the number of the conjugacy classes of G (c.f., Serre [3, p. 32]). Their dimensions satisfy the formula

$$(1) \qquad \sum_{i=1}^{n} (\dim \rho_i)^2 = |G| \ .$$

If G is abelian, then (1) implies that the irreducible representations of G are of dimension 1 (i.e., they are characters) and that their number is equal to $|G|$, which is in this case the number of the conjugacy classes of G. Further, the set of characters of G forms a multiplicative group $\hat{G}$ which is isomorphic to G. If $1 \neq \chi \in \hat{G}$, then we have the following orthogonality relation $\sum_{g \in G} \chi(g) = 0$. A lemma of Artin says that the characters of G are linearly independent, i.e., if a_χ are complex numbers such that $\sum_{\chi \in \hat{G}} a_\chi \chi(g) = 0$ for every $q \in G$, then $a_\chi = 0$ for all $\chi \in G$ (cf. Lang [1, p. 209]). Now, G is canonically isomorphic to the dual $\hat{\hat{G}}$ of $\hat{G}$. Hence, the dual to this lemma is also true: If b_g are complex numbers such that $\sum_{g \in G} b_g \chi(g) = 0$ for every $\chi \in G$, then $b_g = 0$ for all $g \in G$.

If G is again an arbitrary finite group, then we deduce that it has $(G:G')$ characters, where G' is the commutator subgroup of G. Another consequence of formula (1) is that if distinct irreducible representations $\rho_1,\ldots,\rho_n$ of G satisfy $\sum_{i=1}^{n} (\dim \rho_i)^2 = G$, then they are all the representations of G.

Let ρ be a representation of a finite group G. Then V_ρ can be also considered as a module over the group-ring $\mathbb{C}[G]$. If ρ' is an additional representation of G, then we write $(\rho,\rho') = \dim \mathrm{Hom}_{C[G]}(V_\rho, V_{\rho'})$. The form (ρ,ρ') is clearly symmetric and bilinear with respect to direct sums. If both ρ and ρ' are irreducible, then, by a lemma of Schur, $(\rho,\rho') = 1$ if $\rho = \rho'$ and $(\rho,\rho') = 0$ if $\rho \neq \rho'$ (cf. [2, p. 25]). It follows that two arbitrary representations ρ and ρ' are disjoint, i.e., have no common irreducible subrepresentation, if and only if $(\rho,\rho') = 0$. In particular, an irreducible representation ρ appears in a representation ρ', i.e., $\rho \leq \rho'$, if and only if $(\rho,\rho') \neq 0$; indeed (ρ,ρ') is equal to the multiplicity in which ρ appears in ρ'.

Let $\mathrm{End}_{\mathbb{C}[G]} V_\rho = \mathrm{Hom}_{\mathbb{C}[G]}(V_\rho, V_\rho)$. It is an algebra over $\mathbb{C}$ called the Schur algebra. If ρ is irreducible, then $\mathrm{End}_{\mathbb{C}[G]} V_{n\rho}$ is isomorphic to $M_n(\mathbb{C})$, the algebra of all $n\times n$ matrices over $\mathbb{C}$. If $\rho = \oplus n_i \rho_i$ is the canonic decomposition of a representation ρ, then, by Schur's lemma, $\mathrm{End}_{\mathbb{C}[G]} V = \oplus M_{n_i}(\mathbb{C})$. Hence $(\rho,\rho) = \dim \mathrm{End}_{\mathbb{C}[G]} V_\rho = \sum n_i^2$. It follows that ρ has no multiple components, i.e., that $n_i = 1$ for all i, if and only if $\mathrm{End}_{\mathbb{C}[G]} V_\rho$ is commutative. In this case $\dim \mathrm{End}_{\mathbb{C}[G]} V_\rho$ is the number of components of ρ.

Finally consider a vector space V of dimension n over $\mathbb{C}$. Every base $v_1,\dots,v_n$ of V canonically defines an isomorphism $\mathrm{Aut}\,V \cong GL(n,\mathbb{C})$ (= the group of all $n\times n$ invertible matrices over $\mathbb{C}$). If $\rho: G \to \mathrm{Aut}\,V$ is a representation of V, then we define $\chi_\rho(g)$ to be the trace of $\rho(g)$, where $\rho(g)$ is now considered as an element of $GL(n,\mathbb{C})$ via the above isomorphism. Clearly $\mathrm{tr}\,\rho(g)$ does not depend on the choice of the basis $v_1,\dots,v_n$ of V. Hence $\chi_\rho: G \to \mathbb{C}$ is a well defined function, called the <u>character of</u> ρ. It is constant on conjugacy classes. Also $\chi_{\rho_1\oplus\rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$. Therefore χ_ρ is said to be <u>irreducible</u> if ρ is irreducible. If $\dim\rho = 1$, then $\chi_\rho = \chi$. In general one defines $\dim\chi_\rho = \dim\rho$ and refers to χ_ρ as a <u>higher dimensional character</u>.

2. Induced representations.

Let G be a finite group and let H be a subgroup operating on a finite dimensional $\mathbb{C}$-vector space W through a representation $\tau: H \to \mathrm{Aut}\,W$. Define a vector space V to be the set of all functions $f: G \to W$ that satisfy

$$f(hg) = \tau(h)f(g) \quad \text{for all} \quad h \in H \text{ and } g \in G .$$

Thus, in order to define an element $f \in V$, it suffices to give its values on a system of representatives H/G of the left classes of G modulo H. Define an operation of G on V by

$$(sf)(g) = f(gs) \quad \text{for} \quad s,g \in G \text{ and } f \in V .$$

The $\mathbb{C}[G]$ - Module V thus obtained is called the <u>induced module of</u> W <u>from</u> H <u>to</u> G and is denoted by $\mathrm{Ind}_H^G\tau$.

We embed W in V by mapping each $w \in W$ onto the function $f_w: W \to \mathbb{C}$ defined by $f_w(g) = \tau(g)w$ if $g \in H$ and $f_w(g) = 0$ if $g \in G-H$. Clearly this is a $\mathbb{C}[H]$ - modules embedding. The image of W in V consists of all the functions $f \in V$ that vanish on $G-H$.

Let now $G = \bigcup_{r \in R} rH$ be a decomposition of G into left classes modulo H. For every $f \in V$ and for every $r \in R$ we define a function $f_r \in V$ by $f_r(g) = f(g)$ if $g \in Hr^{-1}$ and $f_r(g) = 0$ otherwise. Then $r^{-1}f_r$ belongs to W (after identifying W with its image in V) and $f = \sum_{r \in R} r(r^{-1}f_r)$. Thus V is isomorphic to $\bigoplus_{r \in R} rW$. In particular we have that $\dim V = (G:H) \dim W$. Using this isomorphism one obtains also a canonical isomorphism $V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$, where G operates on the right-hand side by multiplication on the left of the first factor. This form of the induced representation is convenient to prove the following fundamental properties. (a) Transitivity: If J is a subgroup of H and $\tau: J \to \operatorname{Aut} U$ is a representation of J, then

$$\operatorname{Ind}_J^G U = \operatorname{Ind}_H^G (\operatorname{Ind}_J^H U) .$$

(b) Frobenius reciprocity theorem: With the above notation let E be a $\mathbb{C}[G]$-module and denote by $\operatorname{Res}_H^G E$ the $C[H]$-module obtained from E by considering only the action of H. Then we have the following canonical isomorphism:

$$\operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G W, E) \cong \operatorname{Hom}_{\mathbb{C}[H]}(W, \operatorname{Res}_H^G E)$$

(cf. [3, p. 23]). In particular,

$$\dim \operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G W, E) = \dim \operatorname{Hom}_{\mathbb{C}[H]}(W, \operatorname{Res}_H^G E) .$$

If τ and σ are the representations of H and G that correspond to W and E, respectively, then the last equality can be rewritten, in the notation of section 1, as

$$(\operatorname{Ind}_H^G \tau, \sigma)_G = (\tau, \operatorname{Res}_H^G \sigma)_H .$$

In particular, if both τ and σ are irreducible, then the multiplicity of σ in $\operatorname{Ind}_H^G \tau$ is equal to the multiplicity of τ in $\operatorname{Res}_H^G \sigma$.

Finally, if τ is a representation of a subgroup H of a group G and $\rho = \operatorname{Ind}_H^G \tau$, then χ_ρ can be calculated from χ_τ by the following formula

$$\chi_\rho(g) = \frac{1}{|H|}\sum_{r\in G}\tilde{\chi}_\tau(sgs^{-1}) = \sum_{r\in R}\tilde{\chi}_\tau(rgr^{-1}) ,$$

where $\tilde{\chi}_\tau$ is the function on G that vanishes outside H and coincides with χ_τ on H; R is a system of representatives of right classes of G modulo H (cf. [2, p. 72]).

3. The Schur algebra.

Proposition 3.1: Let H and J be subgroups of a finite group G. Let ρ and σ be representations of H and J, respectively. Then $\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G V_\rho, \mathrm{Ind}_J^G V_\sigma)$ is isomorphic to the vector space of all functions $F: G \to \mathrm{Hom}_{\mathbb{C}}(V_\rho, V_\sigma)$ satisfying

(1) $$F(jgh) = \sigma(j)\circ F(g)\circ\rho(h)$$

for all $j \in J$, $g \in G$ and $h \in H$.

Proof: Let $\hat{\rho} = \mathrm{Ind}_H^G\rho$, $\hat{\sigma} = \mathrm{Ind}_J^G\sigma$ and $n = (G:H)$. Denote by F' the vector space of all functions

$$\varphi: G\times G \to \mathrm{Hom}_{\mathbb{C}}(V_\rho, V_\sigma)$$

that satisfy

(2) $$\varphi(jg_1, hg_2) = \sigma(j)\circ\varphi(g_1,g_2)\circ\rho(h)^{-1}$$

for all $j \in J$, $h \in H$ and $g_1, g_2 \in G$. For every $\varphi \in F'$ we define an element $T_\varphi \in \mathrm{Hom}_{\mathbb{C}}(V_{\hat{\rho}}, V_{\hat{\sigma}})$ as follows: If $f \in V_\rho$, then $T_\varphi f: G \to V_\sigma$ is the map defined by

(3) $$(T_\varphi f)(g) = \frac{1}{n}\sum_{r\in G}\varphi(g,r)(f(r)) ;$$

clearly the map $\varphi \to T_\varphi$ is a homomorphism $F' \to \mathrm{Hom}_{\mathbb{C}}(V_{\hat{\rho}}, V_{\hat{\sigma}})$. It is injective. Indeed, suppose that $T_\varphi = 0$. Let $s \in G$, let $v \in V_\rho$ and define a function $f_{sv} \in V_\rho$ by

$$f_{sv}(g) \begin{cases} \rho(h)v & \text{if } g = hs \\ 0 & \text{if } g \notin Hs \,. \end{cases}$$

Then substituting $f = f_{sv}$ in (3) we have by (2) that $\varphi(g,s)v = 0$. Hence $\varphi(g,s) = 0$, i.e., $\varphi = 0$.

The dimension of F' is equal to $(G:H)(G:J)(\dim\rho)(\dim\sigma)$ by (2). This is also the dimension of $\mathrm{Hom}_{\mathbb{C}}(V_{\hat\rho},V_{\hat\sigma})$. Hence T is an isomorphism.

Denote now by F'_G the subspace of all $\varphi \in F'$ such that $T_\varphi \in \mathrm{Hom}_{\mathbb{C}[G]}(V_{\hat\rho},V_{\hat\sigma})$. Clearly $\varphi \in F'_G$ if and only if

(4) $$\sum_{r \in G} \varphi(g,rx^{-1})(f(r)) = \sum_{r \in G} \varphi(gx,r)(f(r))$$

for all $f \in V_{\hat\rho}$. Substituting $f = f_{sv}$ in (4), we have that (4) is equivalent to the condition

(5) $$\varphi(g,rx^{-1}) = \varphi(gs,r) \quad \text{for all } g,r,x \in G \,.$$

For every function $F\colon G \to \mathrm{Hom}_{\mathbb{C}}(V_\rho,V_\sigma)$ that satisfies (1), we define a function $\varphi\colon G\times G \to \mathrm{Hom}_C(V_\rho,V_\delta)$ by

(6) $$\varphi(g_1,g_2) = F(G_1g_2^{-1}) \,.$$

Then φ satisfies (5) and thus it belongs to F'_G. Conversely, starting from φ in F'_G, we define an $F\colon G \to \mathrm{Hom}_{\mathbb{C}}(V_\rho,V_\sigma)$ by

$$F(g) = \varphi(g,1) \,.$$

Then F satisfies (1) and the φ defined by (6) coincides with the one we started with. Thus F is isomorphic to F'_G.

For every $F \in F$ denote by T_F the element of $\mathrm{Hom}_{\mathbb{C}[G]}(V_{\hat\rho},V_{\hat\sigma})$ defined by

(7) $$(T_F f)(g) = \frac{1}{n} \sum_{r \in G} F(gr^{-1})(f(r)) \,.$$

Then the map $F \to T_F$ is the desired isomorphism. //