Mathematics

A MODELING APPROACH

MARVIN L. BITTINGER
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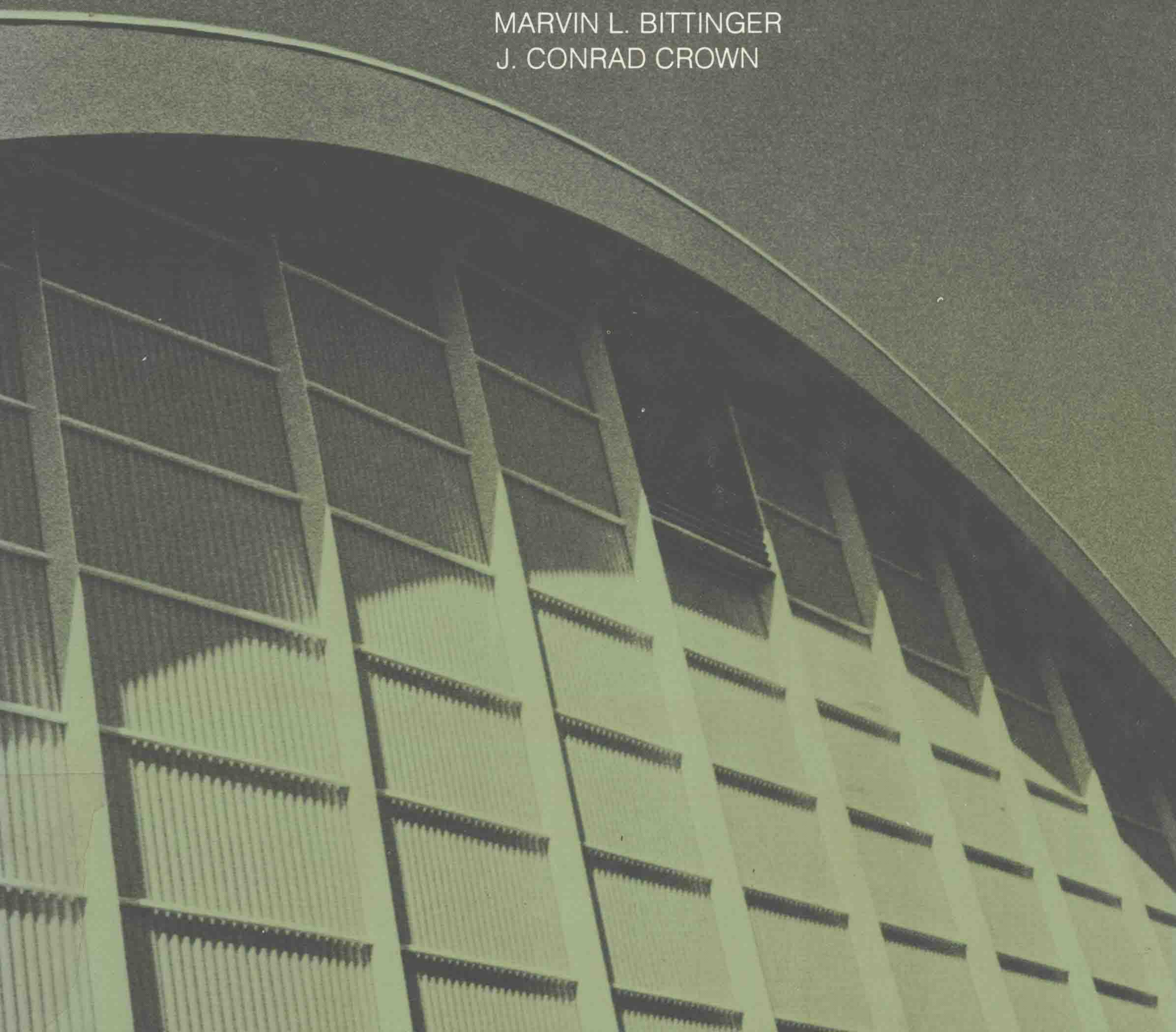

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A MODELING APPROACH

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Preface

The material in this book continues on from basic algebra and introduces the student to areas of Finite Mathematics and Calculus which have applications in business, economics, management, the social and behavioral sciences, and biology. The basic material can be covered in two semester courses.

While much of the material in this book has been taken from the second editions of *Finite Mathematics: A Modeling Approach* by Bittinger and Crown and *Calculus: A Modeling Approach* by Bittinger, many sections have been rewritten after class-testing the original texts, incorporating suggestions from both instructors and students to present the ideas more clearly. Also, by request, a section on "Mathematics of Finance" has been included in the present text.

1. Intuitive approach

While this word has many meanings and interpretations, its use here, for the most part, means "experience-based." That is, when a concept is being taught, the learning is based on the student's prior experience or new experience given before the concept is formalized. For example, in a maximum–minimum problem a function is usually derived which is to be maximized or minimized. Instead of forging ahead with the standard calculus solution, the student is asked to stop and compute some function values. This experience provides the student with more insight into the problem. Not only does the student discover that different dimensions yield different volumes, if volume is to be maximized, but the dimensions which yield the maximum volume might even be conjectured as a result of the calculations. Provision for use of the hand calculator also provides for an intuitive approach (see later comments).

2. Design and format

Each page has an outer margin which is used in several ways. (1) In the margin are sample, developmental, and exploratory exercises, placed with the text material so that the student can become actively involved in the development of the topic. These margin exercises have proved to be extremely beneficial. (2) For each section the objectives are stated in behavioral terms at the top of the page. These can be easily spotted by the student, and when the typical question arises "What material am I responsible for?" these objectives provide an answer. They may also help take the fear out of the word "mathematics."

3. The hand calculator

Exercises in this text can be done with or without a hand calculator. Most students, we find, not only have calculators but assume that calculators are *always* helpful to them. While there are many types of problems for which the calculator can reduce the work of computation, there are also many problems where there are naturally occurring fractions, and automatic conversion of all fractions to decimals may bring more distress than relief. For example, in the solution of systems of linear equations, fractions such as ⅓ have no exact decimal equivalent, and consequently conversion to decimals introduces the problem of "round-off" error. In general we feel that calculators should *not* be used automatically but rather should be reserved for cases where they are necessary (or where tables would be required), or where they reduce the tedium of computation.

4. Applications

Relevant and factual applications are included throughout the text to maintain interest and motivation. Problems in linear programming are of particular interest to students in business and management curricula. Problems in natural growth and decay (involving exponential and logarithmic functions) have applications in almost all areas ranging from birth rate to salvage value. Population growth is considered in the context of several mathematical models, not just exponential (see Chapter 10). When the exponential model is studied, other applications such as continuously compounded interest and the demand for natural resources are also considered. The notions of total revenue, cost, and profit, together with their related derivatives (marginal functions), are threads which run through the text, providing continued reinforcement and unification.

5. Tests

Each chapter ends with a chapter test. We have found that such sample tests supplement the Chapter Objectives and take much of the anxiety out of mathematics since the students know what is expected of them. All the answers to these tests are in the back of the book. Additional alternate forms of these tests appear, classroom-ready, in the *Instructor's Manual,* which is now in 8½″ × 11″ format to facilitate copying.

6. Exercises

Great care has been given to constructing the exercises, most of which support the behavioral objectives. Many of the Linear Programming exercises have been reworked to simplify the calculations and minimize the occurrence of fractions. The first exercises in each set are quite easy, while later ones become progressively more difficult. Most of the exercises are similar to Examples worked out in the preceding section. The exercises are also arranged in matching pairs; that is, each odd-numbered exercise is very much like the one immediately following. The odd-numbered exercises have answers in the book, while the even-numbered exercises have answers in the *Instructor's Manual.* All Margin Exercises have answers in the book. It is recommended that the student do all of these, stopping to do them when the text so indicates.

The authors wish to acknowledge the assistance of the following, whose professional reviews were extremely valuable in the preparation of the book: A. Astromoff (San Francisco State University), Walter A. Coole (Skagit Valley College), Richard M. Crownover (University of Missouri—Columbia), Matt Hassett (Arizona State University), Carl D. Meyer (North Carolina State University), Ann O'Connell (Providence College), John W. Toole (University of Maine at Orono), Francis A. Varrichio (Saint Peter's College), Samuel J. Wiley (LaSalle College), and Thomas J. Woods (Central Connecticut State College).

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Contents

*Indicates an optional section in the sense that subsequent material is not dependent on it.

CHAPTER ONE

World running records can be modeled by linear functions. (*Marshall Henrichs*)

Algebra Review, Functions, and Modeling

1.1 EXPONENTS, MULTIPLYING, AND FACTORING

Exponential Notation

The set of integers is as follows:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

Let us review the meaning of an expression

$$a^n,$$

where n is an integer. The number a above is called the *base* and n is called the *exponent*. When n is larger than 1, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}.$$

In other words, a^n is the product of n factors, each of which is a.

Examples Rename without exponents.

a) $4^3 = 4 \cdot 4 \cdot 4$, or 64
b) $(-2)^5 = (-2)(-2)(-2)(-2)(-2)$, or -32
c) $(1.08)^2 = 1.08 \times 1.08$, or 1.1664
d) $(\frac{1}{2})^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$, or $\frac{1}{8}$

DO EXERCISES 1 THROUGH 4. (EXERCISES ARE IN THE MARGIN.)

We define an exponent of 1 as follows:

$$a^1 = a, \quad \textbf{for any number } a.$$

That is, any number to the first power is that number itself. We define an exponent of 0 as follows:

$$a^0 = 1, \quad \textbf{for any nonzero number } a.$$

That is, any nonzero number a to the 0 power is 1.

Examples Rename without exponents.

a) $(-2x)^0 = 1$ b) $(-2x)^1 = -2x$ c) $(\frac{1}{2})^0 = 1$
d) $e^0 = 1$ e) $e^1 = e$ f) $(\frac{1}{2})^1 = \frac{1}{2}$

DO EXERCISES 5 THROUGH 10.

OBJECTIVES

You should be able to

a) Rename an exponential expression without exponents.
b) Multiply exponential expressions by adding exponents.
c) Divide exponential expressions by subtracting exponents.
d) Raise a power to a power by multiplying exponents.
e) Multiply algebraic expressions.
f) Factor algebraic expressions.
g) Solve applied problems involving the comparison of a power like $(3.1)^2$ with 3^2.
h) Solve applied problems involving compound interest.

Rename without exponents.

1. 3^4 **2.** $(-3)^2$

3. $(1.02)^3$ **4.** $(\frac{1}{4})^2$

Rename without exponents.

5. $(5t)^0$ **6.** $(5t)^1$

7. k^0 **8.** m^1

9. $(\frac{1}{4})^1$ **10.** $(\frac{1}{4})^0$

The meaning of a negative integer as an exponent is as follows:

$$a^{-n} = \frac{1}{a^n}, \quad \textbf{for any nonzero number } a.$$

That is, any nonzero number a to the $-n$ power is the reciprocal of a^n.

Examples Rename without negative exponents.

a) $2^{-5} = \dfrac{1}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \dfrac{1}{32}$

b) $10^{-3} = \dfrac{1}{10 \cdot 10 \cdot 10} = \dfrac{1}{1000}$, or 0.001

c) $(\frac{1}{4})^{-2} = \dfrac{1}{(\frac{1}{4})^2} = \dfrac{1}{\frac{1}{4} \cdot \frac{1}{4}} = \dfrac{1}{\frac{1}{16}} = 1 \cdot \dfrac{16}{1} = 16$

d) $x^{-5} = \dfrac{1}{x^5}$

e) $e^{-k} = \dfrac{1}{e^k}$

f) $t^{-1} = \dfrac{1}{t^1} = \dfrac{1}{t}$

DO EXERCISES 11 THROUGH 17.

Properties of Exponents

Note the following:

$$\begin{aligned} b^5 \cdot b^{-3} &= (b \cdot b \cdot b \cdot b \cdot b)\frac{1}{b \cdot b \cdot b} \\ &= \frac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b \cdot b} \\ &= \frac{b \cdot b \cdot b}{b \cdot b \cdot b} \cdot b \cdot b \\ &= 1 \cdot b \cdot b \\ &= b^2. \end{aligned}$$

The result could have been obtained by adding the exponents. This is true in general.

For any number a, and any integers n and m,

$$a^n \cdot a^m = a^{n+m}.$$

(To multiply when the bases are the same, add the exponents.)

Rename without negative exponents.

11. 2^{-4} **12.** 10^{-2}

13. $(\frac{1}{4})^{-3}$ **14.** t^{-7}

15. e^{-t} **16.** M^{-1}

17. $(x + 1)^{-2}$

Multiply.

18. $t^4 \cdot t^5$

19. $t^{-4} \cdot t$

20. $10e^{-4} \cdot 5e^{-9}$

21. $t^{-3} \cdot t^{-4} \cdot t$

22. $4b^5 \cdot 6b^{-2}$

Divide.

23. $\dfrac{x^6}{x^2}$

24. $\dfrac{x^2}{x^6}$

25. $\dfrac{e^t}{e^t}$

26. $\dfrac{e^2}{e^k}$

27. $\dfrac{e^5}{e^{-7}}$

28. $\dfrac{e^{-5}}{e^{-7}}$

Examples Multiply.

a) $x^5 \cdot x^6 = x^{5+6} = x^{11}$

b) $x^{-5} \cdot x^6 = x^{-5+6} = x$

c) $2x^{-3} \cdot 5x^{-4} = 10x^{-3+(-4)} = 10x^{-7}$

d) $r^2 \cdot r = r^{2+1} = r^3$

DO EXERCISES 18 THROUGH 22.

Note the following:

$$b^5 \div b^2 = \frac{b^5}{b^2} = \frac{b \cdot b \cdot b \cdot b \cdot b}{b \cdot b} = \frac{b \cdot b}{b \cdot b} \cdot b \cdot b \cdot b = 1 \cdot b \cdot b \cdot b = b^3.$$

The result could have been obtained by subtracting the exponents. This is true in general.

For any nonzero number a and any integers n and m,

$$\frac{a^n}{a^m} = a^{n-m}.$$

(To divide when the bases are the same, subtract the exponents.)

Examples Divide.

a) $\dfrac{a^3}{a^2} = a^{3-2} = a^1 = a$

b) $\dfrac{x^7}{x^7} = x^{7-7} = x^0 = 1$

c) $\dfrac{e^3}{e^{-4}} = e^{3-(-4)} = e^{3+4} = e^7$

d) $\dfrac{e^{-4}}{e^{-1}} = e^{-4-(-1)} = e^{-4+1} = e^{-3}$, or $\dfrac{1}{e^3}$

DO EXERCISES 23 THROUGH 28.

Note the following:

$$(b^2)^3 = b^2 \cdot b^2 \cdot b^2 = b^{2+2+2} = b^6.$$

The result could have been obtained by multiplying the exponents. This is true in general.

For any number a, and any integers n and m,

$$(a^n)^m = a^{nm}.$$

(To raise a power to a power, multiply the exponents.)

Examples Simplify.

a) $(x^{-2})^3 = x^{-2 \cdot 3} = x^{-6}$, or $\dfrac{1}{x^6}$

b) $(e^x)^2 = e^{2x}$

c) $(3x^3y^4)^2 = 3^2(x^3)^2(y^4)^2 = 9x^6y^8$

d) $(2x^4y^{-5}z^3)^{-3} = 2^{-3}(x^4)^{-3}(y^{-5})^{-3}(z^3)^{-3}$

$$= \frac{1}{2^3}x^{-12}y^{15}z^{-9}, \text{ or}$$

$$= \frac{y^{15}}{8x^{12}z^9}$$

DO EXERCISES 29 THROUGH 33.

Multiplication

The distributive laws are important in multiplying. The laws are as follows:

For any numbers *a*, *b*, and *c*,

$$a(b + c) = ab + ac, \quad \text{and} \quad a(b - c) = ab - ac.$$

Examples Multiply.

a) $3(x-5) = 3 \cdot x - 3 \cdot 5 = 3x - 15$

b) $P(1+i) = P \cdot 1 + P \cdot i = P + Pi$

c) $(x - 5)(x + 3) = (x - 5)x + (x - 5)3$

$$= x \cdot x - 5x + 3x - 5 \cdot 3$$

$$= x^2 - 2x - 15$$

d) $(a + b)(a + b) = (a + b)a + (a + b)b$

$$= a \cdot a + ba + ab + b \cdot b$$

$$= a^2 + 2ab + b^2$$

DO EXERCISES 34 THROUGH 38.

The following formulas, which are obtained using the distributive laws, are useful in multiplying.

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (1)$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad (2)$$

$$(a - b)(a + b) = a^2 - b^2 \quad (3)$$

Simplify.

29. $(x^{-4})^3$ **30.** $(e^2)^2$

31. $(e^x)^3$ **32.** $(5x^3y^5)^2$

33. $(4x^{-5}y^{-6}z^2)^{-4}$

Multiply.

34. $2(x + 7)$ **35.** $P(1 - i)$

36. $(x - 4)(x + 7)$ **37.** $(a - b)(a - b)$

38. $(a - b)(a + b)$

Multiply.

39. $(x - h)^2$ **40.** $(3x + t)^2$

41. $(5t - m)(5t + m)$

Factor.

42. $P - Pi$

43. $x^2 + 10xy + 25y^2$

44. $4x^2 + 28x + 40$

45. $25c^2 - d^2$

46. $3x^2h + 3xh^2 + h^3$

47. How close is $(5.1)^2$ to 5^2?

Examples Multiply.

a) $(x + h)^2 = x^2 + 2xh + h^2$
b) $(2x - t)^2 = (2x)^2 - 2(2x)t + t^2 = 4x^2 - 4xt + t^2$
c) $(3c + d)(3c - d) = (3c)^2 - d^2 = 9c^2 - d^2$

DO EXERCISES 39 THROUGH 41.

Factoring

Factoring is the reverse of multiplication. That is, to factor an expression, we find an equivalent expression that is a product. Always remember to look first for a common factor.

Examples Factor.

a) $P + Pi = P \cdot 1 + P \cdot i = P(1 + i)$ (We used a distributive law.)
b) $2xh + h^2 = h(2x + h)$
c) $x^2 - 6xy + 9y^2 = (x - 3y)^2$
d) $x^2 - 5x - 14 = (x - 7)(x + 2)$ (Here we looked for factors of -14 whose sum is -5.)
e) $x^2 - 9t^2 = (x - 3t)(x + 3t)$ (We used $(a - b)(a + b) = a^2 - b^2$.)

DO EXERCISES 42 THROUGH 46.

In later work we will consider expressions like

$$(x+h)^2 - x^2.$$

To simplify this, first note that

$$(x + h)^2 = x^2 + 2xh + h^2.$$

Subtracting x^2 on both sides of this equation, we get

$$(x + h)^2 - x^2 = 2xh + h^2.$$

Factoring out an h on the right side we get

$$(x + h)^2 - x^2 = h(2x + h). \tag{4}$$

Let us now use this result to compare two squares.

Example How close is $(3.1)^2$ to 3^2?

Solution Substituting $x = 3$ and $h = 0.1$ in equation (4) we get

$$(3.1)^2 - 3^2 = 0.1(2 \cdot 3 + 0.1) = 0.1(6.1) = 0.61.$$

So $(3.1)^2$ differs from 3^2 by 0.61.

DO EXERCISE 47.

*(Optional) Compound Interest

Suppose we invest P dollars at interest rate i, compounded annually. The amount A_1 in the account at the end of 1 year is given by

$$A_1 = P + Pi = P(1 + i) = Pr,$$

where, for convenience,

$$r = 1 + i.$$

Going into the second year we have Pr dollars, so by the end of the second year we would have the amount A_2 given by

$$A_2 = A_1 \cdot r = (Pr)r = Pr^2.$$

Going into the third year we have Pr^2 dollars, so by the end of the third year we would have the amount A_3 given by

$$A_3 = A_2 \cdot r = (Pr^2)r = Pr^3.$$

In general,

If an amount P is invested at interest rate i, compounded annually, in t years it will grow to the amount A given by

$$\mathbf{A = P(1 + i)^t.}$$

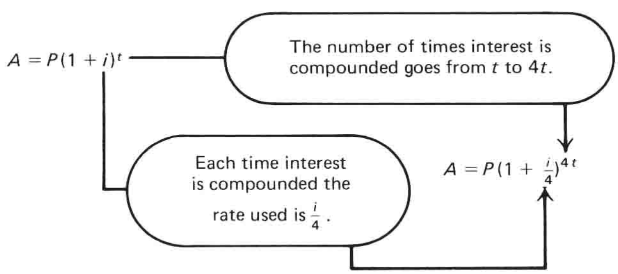
Example 1 Suppose \$1000 is invested at 8% compounded annually. How much is in the account at the end of 2 years?

Solution We substitute into the equation $A = P(1 + i)^t$ and get

$$A = 1000(1 + 0.08)^2 = 1000(1.08)^2 = 1000(1.1664) = \$1166.40.$$

DO EXERCISE 48.

If interest is compounded quarterly, we can find a formula like the one above as follows:



48. Suppose \$1000 is invested at 7% compounded annually. How much is in the account at the end of 2 years?

* This will be reconsidered in Chapter 8, "Mathematics of Finance."