Topological Methods for Set-Valued Nonlinear Analysis

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To

Our beloved professor Dr. Eneyet Ullah Tarafdar, his wife — Mrs. Sangha Mitra, his daughter — Hashina Tarafdar and his two sons — Abir Tarafdar and Ashique Tarafdar.

Preface

This book is a monograph of a significant and recent publications in non-linear analysis involving set-valued mappings. A map $T : X \to 2^Y$ is said to be a set-valued mapping if for each $x \in X$, $T(x) \subset Y$.

We need analysis, topology and geometry, i.e., a mixture of these three fields, in studying the theory of set-valued mappings. There have been a significant number of publiactions in this area of research over the last 40 years. These have become possible because there are huge applications in the fileds of Physics, Biology, Control Theory, Optimization, Economics and Game Theory.

We shall cover the following topics in this book: contraction mappings, fixed point theorems, minimax inequalities, end points, variational inequalities, generalized variational inequalities, and generalized quasi-variational inequalities, equilibrium analysis in economics, best approximation and fixed point theorems, topological degree theory,and non-expansive types of mappings and fixed point theorems.

In Chapter 5, we shall present variational inequalities, quasi-variational equalities and equilibrium analysis in economics. We have applied the topological methods to study the equilibrium analysis in economics. We shall discuss them in more details in the Introduction Chapter. In Chapter 6, we shall discuss best approximation and fixed point theorems for set-valued mappings in topological vector spaces. Finally, in Chapters 7 and 8 we shall present some aspects of degree theories for set-valued mappings and non-expansive types of mappings and fixed point theorems in locally convex topological vector spaces.

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After the sudden and unfortunate death of Dr. Enayet Tarafdar in November, 2002, I continued with the project and tried to finish it with the help of my other friends and well wishers who were working in this area of research. In this direction, I would like to mention the names of Dr. George Yuan and Dr. Peter Watson who tried to help me in finishing this project with their valuable suggestons and inputs.

Thanks goes to Dr. Yuan for partially completing the Promotional Questionnaire of this book before its publication.

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October 10, 2007

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Chapter 1

Introduction

Our main objective in this book is to study some aspects of non-linear analysis which involve set-valued mappings. However, a single valued mapping $T : X \to Y$ of a non-empty set X into a non-empty set Y can be regarded as a set-valued mapping by considering one point $\{T(x)\}$ for each each $x \in X$.

The various aspects of fixed points, minimax inequalities, end points, variational inequalities, generalized variational inequalities, and generalized quasi-variational inequalities, equilibrium analysis in economics, best approximation and fixed point theorems, topological degree theory,and non-expansive types of mappings and fixed point theorems, and related topics are considered in this book.

It is well known that fixed point theory is very important in mathematics. The close relationship between fixed point theory and mathematical economics can be illustrated in many ways. The usefullness of Brouwer's fixed point theorem was recognized by John Von Neumann when he developed the foundations of game theory in 1928.

Fixed point and coincedence theorems for set-valued mappings and their applications to minimax theorems and economics originated from the works of John Von Neumann (Neumann (1937)) (see also Neumann (1928b), Neumann (1928a), Neumann and Morgenstern (1944) and Neumann and Morgenstern (1947)). Then the theory was advanced by Kakutani (1941), Fan (1952) and others (see Zeidler's book (Zeidler (1985))).

In most of the economic papers appearing in any journals of economics, one can find the terms economic equilibria, Pareto optimum in abundance. Pareto talked about the optimum which has come to be known popularly as Pareto Optimum (Pareto allocation). In the last century, new discipline called mathematical economics — has evolved into a highly developed and fast growing branch of mathematics blended with the components of economy, games, econometrics, psychology and many related areas.

In fact, in a fascinating article Franklin (1983) (incidentally has a book, see Franklin (1980)) wrote: In 1969 a spokesman for the Nobel foundation welcomed the new prize subject, economics, as "the oldest of the arts, the youngest of the sciences". It might be fair to say that economics became a science when it started making significant use of mathematics.

In this book, we have applied the topological methods to study the equilibrium analysis in economics, i.e., to prove the existence of equilibrium of social economics. It seems that in this area nothing dominates more significantly than fixed point theory of set-valued mappings. In fact, Nobel laureate Debreu (1959) proved two fundamental theorems of mathematical economics by using Kakutani's fixed point theorem.

Let E be a topological vector space and A a non-empty subset of E. If $S, T : A \to 2^E$ are correspondences, then $T \cap S : A \to 2^E$ is a correspondences defined by $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$.

Ding, Kim and Tan introduced the notions of correspondences of class $\mathcal{L}_\theta^*$, $\mathcal{L}_\theta^*$-majorant of ϕ at x and $\mathcal{L}_\theta^*$- majorized correspondences in Ding, Kim, and Tan (1992) as follows:

Let X be a topological space, Y be a non-empty subset of a vector space $E, \theta : X \to E$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then (1) ϕ is said to be of class $\mathcal{L}_\theta^*$ if for every $x \in X$, $con\phi(x) \subset Y$ and $\theta(x) \notin con\phi(x)$ and for each $y \in Y$, $\phi_{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X; (2) a correspondence $\phi_x : X \to 2^Y$ is said to be an $\mathcal{L}_\theta^*$ majorant of ϕ at x if there exists an open neighborhood N_x, of x in X such that (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z))$ and $\theta(z) \notin con\phi_x(z)$ (b) for each $z \in X$, $con\phi_x,(z) \subset Y$ and (c) for each $y \in Y, \phi_1(y)$ is open in X; (3) ϕ is $\mathcal{L}_\theta^*$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $\mathcal{L}_\theta^*$-majorant of ϕ at x.

In view of Yannelis and Prabhakar (1983, p. 239, Lemma 5.1), Ding, Kim and Tan's notions of the correspondence ϕ being of class $\mathcal{L}_\theta^*$ or $\mathcal{L}_\theta^*$-majorized generalize the notions $\phi \in \mathcal{C}(\mathcal{X}, \mathcal{Y}, \theta)$ or $\mathcal{C}$-majorized respectively which were introduced by Tulcea (1986, p. 2). Ding, Kim and Tan pointed out that their map $\theta : X \to E$ is less restrictive than that of [Tulcea (1986)], where $\theta : X \to Y$. In most applications, either (I) X and Y are non-empty subsets of the same topological vector space E and $\theta(x) = x$ for all $x \in X$, or (II) $X = \Pi_{i=I} X_I$ and $\theta(x) = \pi_j(x)$ for all $x \in X$, where $\pi_j : X \to X_j$ is the projection of X onto X_j and X_j and Y are non-empty subsets of the same topological vector space E.

Ding, Kim and Tan observed that when $X = Y$ and is convex (and $\theta(x) = x$ for all $x \in X$), the notion of correspondence of class $\mathcal{L}_\theta^*$ coincides with the notion of correspondence of class $\mathcal{L}$ introduced by [Yannelis and Prabhakar (1983)] and the notions of $\mathcal{L}_\theta^*$-majorant of ϕ at x and $\mathcal{L}_\theta^*$-majorized correspondence generalize the notions of $\mathcal{L}$-majorant of ϕ at x and $\mathcal{L}$-majorized correspondence respectively also introduced by [Yannelis and Prabhakar (1983)]. In the special case (I), where $\theta = 1_x$, the identity map on X or (II), where $\theta = \pi_j$, $\mathcal{L}^*$ is written in place of $\mathcal{L}_\theta^*$ if there is no ambiguity.

It should be noted that if ϕ is $\mathcal{L}_\theta{}^*$-majorized, then for $x \in X, \theta(x) \notin con\phi(x)$ and $con\phi(x) \subset Y$.

Let I be a (possibly infinite) set of agents. For each agent $i \in I$, let its choice set or strategy set X_i be a non-empty set in a topological vector space. Let $X = \Pi_{i\in I}X_I$. If $i \in I$, let $\pi_i : X \to X_i$ be the projection of X onto X_I and for $x \in X$, let x_i denote the projection $\pi_i(x)$ of x on X_i. Let $P_i : X \to 2^{x_i}$ be an *irreflexive* preference correspondence, i.e., $x_i \notin P_i(x)$ for all $x \in X$. Following [Gale and Mas-Colell (1978)], the collection $(X_i, P_i)_{i\in I}$ will be called a *qualitative game.* A point $\hat{x} \in X$ is said to be an *equilibrium* of that game if $P_i(\hat{x}) = \emptyset$ for all $i \in I$. For each $i \in I$, let A_i be a non-empty subset of X; if $i \in I$ is arbitrarily fixed, we define

$$\Pi_{j\neq I, j\in i}A_I \oplus A_I = \{x = (x_k)_{k\in I} \in X : x_k \in A_k \text{ for each } k \in I\}$$

Let I be a (finite or an infinite) set of agents. An *abstract economy* $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i), where X_i is a topological space, $A_i : \Pi_{j\in I}X_j \to 2^{X_i}$ and $B_i\Pi_{j\in I}X_j \to 2^{X_i}$ are constraint correspondences and $P_i\Pi_{j\in I}X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\hat{x} \in X = \Pi_{i\in I}X_i$ such that for each $i \in I, \hat{x}_i \in clB_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. When $A_i = B_i$ for each $i \in I$, our definitions of an abstract economy and an equilibrium coincide with the standard definitions; e.g., in Borglin and Keiding (1976, p. 315), or in Yannelis and Prabhakar (1983, p. 242).

In the following chapters, if E is a topological vector space, we shall denote the dual space of E, i.e. the vector space of all continuous linear functionals on E, by E^* and the pairing between E^* and E by $\langle w, x\rangle$ for each $w \in E^*$ and $x \in E$, and by $Re\langle w, x\rangle$ the real part of the pairing between $w \in E^*$ and $x \in E$. Unless otherwise stated, if A is a subset of E, we shall denote by 2^A the family of all non-empty subsets A and by clA the closure in E, and by coA, the convex hull of A. Also, we shall denote by $\mathcal{F}(\mathcal{A})$ the family of all non-empty finite subsets of A, by $\mathbb{R}$ the set of all real numbers and $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$.

Let E be a topological vector space. For each $x_0 \in E$, each non-empty subset A of E and each $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in E^* : |\langle y, x_0\rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in E^* : \sup_{x\in A} |\langle y, x\rangle| < \epsilon\}$. Let $\sigma\langle E^*, E\rangle$ be the topology on E^* generated by the family $\{W(x; \epsilon) : x \in E \text{ and } \epsilon > 0\}$ as a sub-base for the neighbourhood system at 0 and $\delta\langle E^*, E\rangle$ be the topology on E^* generated by the family $\{U(A; \epsilon) : A$ is a non-empty bounded subset of E and $\epsilon > 0\}$ as a base for the neighbourhood system at 0. We note that E^*, when equipped with the topology $\sigma\langle E^*, E\rangle$ or the topology $\delta\langle E^*, E\rangle$, becomes a locally convex Hausdorff topological vector space. Furthermore, for a net $\{y_\alpha\}_{\alpha\in\Gamma}$ in E^* and for $y \in E^*$, (i) $y_\alpha \to y$ in $\sigma\langle E^*, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta\langle E^*, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E. The topology $\sigma\langle E^*, E\rangle$ (respectively, $\delta\langle E^*, E\rangle$) is called the *weak* topology* (respectively, the *strong topology*) on E^*. If $p \in E$, $\hat{p}$ is the linear functional on E^* defined by $\hat{p}(f) = f(p)$ for each $f \in E^*$.

Let X be a non-empty subset of E. Then X is a *cone* in E if X is convex and $\lambda X \subset X$ for all $\lambda \geq 0$. If X is a cone in E, then $\widehat{X} = \{w \in E^* : Re\langle w, x\rangle \geq 0 \text{ for all } x \in X\}$ is also a cone in E^*, called the *dual cone* of X.

We shall now state a result of S. C. Fang (e.g. see [Chan and Pang (1982)] and [Shih and Tan (1986), p. 59]) with a little modification, as follows, made in Lemma 2.4.2 in Tan (1994):

Lemma 1.1 *Let X be a cone in a Hausdorff topological vector space E and $T : X \to 2^{E^*}$ be a map. Then the following statements are equivalent:*

(a) There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in X$.

(b) There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y}\rangle = 0$ and $\hat{w} \in \widehat{X}$.

Let $y \in E$. Then the *inward set* of y with respect to X is the set $I_X(y) = \{x \in E : x = y + r(u - y)$ for some $u \in X$ and $r > 0\}$. We shall denote by $\overline{I_X(y)}$ the closure of $I_X(y)$ in E.

Let X and Y be topological spaces and $T : X \to 2^Y$. Then T is said to be:

upper (respectively, *lower*) *semicontinuous* at $x_0 \in X$ [Berge (1963), p. 109] if for each open set G in Y with $T(x_0) \subset G$ (respectively, $T(x_0) \cap G \neq \emptyset$), there exists an open neighbourhood U of x_0 in X such that $T(x) \subset G$ (respectively, $T(x) \cap G \neq \emptyset$) for all $x \in U$;

upper (respectively, *lower*) *semicontinuous* on X if T is upper (respectively, lower) semicontinuous at each point of X.

Moreover, T is said to be continuous on X if it is both upper semi-continuous and lower semi-continuous on X.

Let X be a non-empty subset of E and $T : X \to 2^{E^*}$. Then T is said to be:

(*i*) *monotone* (on X)[Browder (1976), p. 79] if for each $x, y \in X$, each $u \in T(x)$ and each $w \in T(y)$, $Re\langle w - u, y - x\rangle \geq 0$;

(*ii*) *semi-monotone* [Bae, Kim, and Tan (1993), pp. 236–237] (on X) if for each $x, y \in X$, $\inf_{u \in T(x)} Re\langle u, y - x\rangle \leq \inf_{w \in T(y)} Re\langle w, y - x\rangle$.

It is clear that if T is monotone, then T is semi-monotone. The converse is in general false, see Example 2 in [Bae et al. (1993)].

A real-valued function $\psi : X \to \mathbb{R}$ defined on a convex subset X of E is said to be *quasi-concave* if for every real number α the set $\{x \in X : \psi(x) > \alpha\}$ is convex.

If X is a topological space and $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an open cover for X, then a partition of unity subordinated to the open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a family $\beta_\alpha : \alpha \in \mathcal{A}\}$ of continuous real-valued functions $\beta_\alpha : X \to [0, 1]$ such that

(1) $\beta_\alpha(y) = 0$ for all $y \in X \backslash U_\alpha$,

(2) {support $\beta_\alpha : \alpha \in \mathcal{A}\}$ is locally finite and

(3) $\Sigma_{\alpha \in \mathcal{A}} \beta_\alpha(y) = 1$ for each $y \in X$.

Let X be a non-empty subset of a topological vector space E and $T : X \to 2^{E^*}$ be a map. Then the generalized variational inequality problem associated with X and T is to find $\hat{y} \in X$ such that generalized variational inequality $\sup_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leq 0$ for all $x \in X$ holds, or to find $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in X$ holds. When T is single-valued, a generalized variational inequality is called a variational inequality. Browder (1965b) and Hartman and Stampacchia