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AMS-IMS-SIAM Joint Summer Research Conference
July 3–7, 2005
Snowbird, Utah

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Control Methods in PDE-Dynamical Systems

Preface

This volume contains selected papers that were presented at the AMS-IMS-SIAM Joint Summer Research Conference on "Control Methods in PDE-Dynamical Systems," held at the Snowbird Resort, Utah, July 3–7, 2005. This conference was conceived and proposed by the underwriters in February 2004, with one overriding aim: to remain rooted in the topic of controlled PDE systems while reaching out to an ostensibly distinct, yet scientifically related, research community in mathematics; namely, those researchers involved in the study of dynamical properties and asymptotic long-time behavior (in particular, stability) of PDE-mixed problems. It is this community of PDE-based dynamical system specialists that the conference sought to bring together with the community of PDE-control and optimization theorists. These two groups have been concentrated in (roughly) complementary research areas; both in terms of the types of PDEs under investigation and of the nature of the questions asked.

Indeed, the PDE control group—while not neglecting parabolic PDEs—has predominantly been focused on more challenging (non-smoothing) hyperbolic or hyperbolic-like (Petrowski-type) PDEs, typically endowed with low regularity properties. For these dynamics, this group has studied issues such as: (i) optimal (global) interior and boundary (trace) regularity of mixed (initial and boundary value) problems (that is, the issue of well-posedness); (ii) global exact controllability and, equivalently, by duality, corresponding continuous observability estimates (of inverse-type); (iii) uniform stabilization of original conservative (energy preserving) problems: global in the linear case, or local and global in the nonlinear case, either by the insertion of suitable damping or dissipation, or else through the introduction of optimization theory; (iv) well-posedness, regularity, or blow-up of *finite energy* solutions, corresponding to feedback *controlled* nonlinear problems. Here, feedback dissipative mechanisms that are effective in securing good stability of hyperbolic dynamics are typically "rough" or "unbounded." Therefore, the analysis of the resulting nonlinear feedback problem is typically outside the realm of perturbation theory and requires very special considerations rooted in nonlinear PDE theory (e.g., weak convergence methods, compensated compactness, etc.); (v) control-theoretic properties of controllability, asymptotic behaviour and optimality for weak solutions of (hyperbolic) conservation laws and balance laws.

The second, dynamical system, group has dealt mostly with the asymptotic question of long-time behavior of PDEs (non-necessarily dissipative) of smoothing and regularizing parabolic PDEs; and the consequent issues concerning existence of global attractors, their geometric, topological, and structural properties, as well as their dimension (when this is finite).

While pursuing separate interests in their respective range of action with a different focus, and often with a different array of technical tools, the two communities do share, however, a substantial body of common knowledge and background in evolution equations. Thus, it was the organizers' firm conviction that the time was ripe and the momentum propitious to bring them together at a joint conference, to mutually stimulate each other and to share recent advances and breakthroughs in their respective disciplines. These would then serve as springboards for new progress through their combination. This conviction was further buttressed by recent discoveries that certain nontrivial energy methods, initially devised for control-theoretic *a-priori* estimates, once combined with dynamical systems techniques, yield entirely new asymptotic results on well-established, nonlinear PDE systems, particularly hyperbolic and Petrowski-type PDEs.

These expectations are now particularly well reflected in the contributions to this volume. They involve nonlinear parabolic, as well as hyperbolic, equations and their attractors, aeroelasticity, elastic systems, Euler-Korteweg models, thin-film equations, Schrödinger equations, beam equations, variational principles, etc. In addition, the static topics of Helmholtz equations, and Morrey potentials are also prominently featured. A special component of the present volume focuses on hyperbolic conservation laws where, thanks to recent, major theoretical advances, a general mathematical theory is now in place. This is also suitable for the analysis of boundary or distributed control problems, such as they are motivated by various applications including traffic flow models, gas dynamics, etc.

In all of these areas, the reader will find state-of-the-art accounts as stimulating starting points for further research.

The organizers are grateful to all participants for their contributions to the Conference, either by lecturing, by publishing in the present *Proceedings*, or by actively taking part in the intellectual debate at the Conference.

Very warm thanks are extended to the AMS staff; in particular, to Ms. Donna Salter and Ms. Lori Melucci, whose much appreciated efforts and smooth, professional coordination of a large variety of activities were essential to the success of the Conference.

Finally, we wish to thank Ms. Christine Thivierge from the AMS Publication Office, for precious help in connection with the publication of the present volume.

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Asymptotic Stabilization of Systems of Conservation Laws by Controls Acting at a Single Boundary Point

Fabio Ancona and Andrea Marson

ABSTRACT. We establish a general result on the asymptotic stabilization of a strictly hyperbolic system of conservation laws near an equilibrium state by means of controls acting only at one boundary point.

1. Introduction

Consider an $n \times n$ system of conservation laws in one-space variable

$$\partial_t u(t,x) + \partial_x F(u(t,x)) = 0\,, \tag{1.1}$$

on the domain

$$\mathbb{D} \doteq \{(t,x) \in \mathbb{R}^2 \mid t > 0,\, x \in \mathbb{D}_t\}\,, \qquad \mathbb{D}_t \doteq \left]\psi_0(t),\, \psi_1(t)\right[\,. \tag{1.2}$$

Here $u = u(t,x) = \big(u_1(t,x), \dots, u_n(t,x)\big)$ is the vector of the *conserved quantities*, and the components of the vector valued function $F(u) = \big(F_1(u), \dots, F_n(u)\big)$ are the corresponding *fluxes*, while $\psi_0(t), \psi_1(t)$ denote the boundary profiles. We assume that the flux function F is a smooth map defined on an open set $\Omega \subseteq \mathbb{R}^n$, and that the system (1.1) is strictly hyperbolic, i.e. that for all $u \in \Omega$ the Jacobian matrix $DF(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_n(u) \qquad \forall\, u \in \Omega\,. \tag{1.3}$$

We denote by $r_1(u), \dots, r_n(u)$ a corresponding basis of right eigenvectors. Each characteristic field is supposed to be either genuinely nonlinear or linearly degenerate in the sense of Lax [**La**], i.e. for each $i = 1, \dots, n$ one of the following two conditions holds

$$D\lambda_i(u) \cdot r_i(u) > 0 \qquad \forall\, u \in \Omega\,, \tag{1.4}$$

$$D\lambda_i(u) \cdot r_i(u) = 0 \qquad \forall\, u \in \Omega\,. \tag{1.5}$$

We shall also assume that the boundaries of the domain $\mathbb{D}$ are Lipschitz continuous maps $\psi_0, \psi_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that satisfy $\psi_0(t) < \psi_1(t)$ for all $t \geq 0$, and that all

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characteristic speeds are bounded away from their slopes, i.e. that are there exists a fixed integer $p \in \{1, \dots, n\}$, and some constant $c > 0$, so that

$$\begin{aligned} \lambda_p(u) + c &\le \min\left\{\dot\psi_0(t),\, \dot\psi_1(t)\right\} \\ \lambda_{p+1}(u) - c &\ge \max\left\{\dot\psi_0(t),\, \dot\psi_1(t)\right\} \end{aligned} \qquad \text{for a.e.} \quad t \ge 0\,, \qquad \forall\, u \in \Omega\,. \tag{1.6}$$

Because of (1.3), (1.6), for a solution $u(t,x)$ defined on the domain $\mathbb{D}$ in (1.2) there will be $n-p$ characteristic lines entering $\mathbb{D}$ at the boundary $x = \psi_0(t)$, and p characteristic lines entering $\mathbb{D}$ at the boundary $x = \psi_1(t)$. The mixed initial-boundary value problem is thus well posed if we prescribe $n-p$ scalar conditions at $x = \psi_0(t)$, and p scalar conditions at $x = \psi_1(t)$. One can express these boundary conditions in a general form as

$$\begin{cases} b_0\big(u(t,\psi_0(t))\big) = \gamma_0(t)\,, \\ b_1\big(u(t,\psi_1(t))\big) = \gamma_1(t)\,, \end{cases} \qquad t > 0\,, \tag{1.7}$$

where $\gamma_0 : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n-p}$, $\gamma_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}^p$ are (locally) integrable boundary data, while $b_0 : \Omega \to \mathbb{R}^{n-p}$, $b_1 : \Omega \to \mathbb{R}^p$ are suitable smooth maps satisfying the rank conditions

$$\begin{aligned} &\operatorname{rank}\big([Db_0(u)\cdot r_{p+1}(u) \mid \cdots \mid Db_0(u)\cdot r_n(u)]\big) = n-p\,, \\ &\operatorname{rank}\big([Db_1(u)\cdot r_1(u) \mid \cdots \mid Db_1(u)\cdot r_p(u)]\big) = p\,, \end{aligned} \qquad \forall\, u \in \Omega\,, \tag{1.8}$$

which guarantee the well-posedness of the corresponding initial-boundary value problem for (1.1). Here, and throughout the paper, $u(t,\psi_\iota(t))$, $\iota = 0,1$, must be understood as the inner trace of the function $u(t,x)$ along the time-like curve $x = \psi_\iota(t)$.

In the present paper we are concerned with the effect of the boundary conditions on the solution of (1.1) from the point of view of control theory. Namely, given a fixed initial condition

$$u(0,x) = \phi(x) \qquad x \in \,]\psi_0(0),\, \psi_1(0)[\,, \tag{1.9}$$

we assume that the evolution of the system can be affected by an external controller acting through the boundary conditions, and we study the family of states that can be attained, at least asymptotically, by the resulting solution. For example, we may consider a gas confined in a cylinder with a moving piston at its top, ruled (in the one-space dimensional setting) by the classical equations of isentropic gas dynamics, the so-called p-system, that in Lagrangian coordinates reads

$$\begin{cases} \partial_t v - \partial_x w = 0 \\ \partial_t w + \partial_x p(v) = 0\,, \end{cases} \qquad t \ge 0\,, \quad x \in \,]0,h[\,. \tag{1.10}$$

Here, v denotes the specific volume, that is the inverse of the density, w is the velocity of gas, and $p(v)$ denotes the pressure, where $p' < 0$, $p'' > 0$, while h is the height of the cylinder. We assume that we can exert a control on the speed of the piston, which corresponds to consider the input control $\alpha(t)$ acting at the boundary point $x = h$ by means of the boundary condition

$$w(t,h) = \alpha(t) \qquad t > 0\,, \tag{1.11}$$

while the velocity of the gas is zero on the bottom of the cylinder, which yields at $x = 0$ the boundary condition

$$w(t,0) = 0 \qquad t > 0\,. \tag{1.12}$$

Once is given an initial data

$$\begin{aligned} v(0,x) &= \overline{v}(x)\,, \\ w(0,x) &= \overline{w}(x)\,, \end{aligned} \qquad x \in \,]0,h[\,, \tag{1.13}$$

and any nearby constant state $(v^*, 0)$, a natural question is wether we can find a boundary control $\alpha(t)$ so that the corresponding solution $t \mapsto (w(t,\cdot), v(t,\cdot))$ of (1.10)-(1.13) approaches $(v^*, 0)$ as $t \to \infty$. If this problem has a positive answer we will say that the system (1.10), with boundary conditions (1.11)-(1.12), is *asymptotically boundary stabilizable* near any constant state $(v^*, 0)$.

Aim of the present paper is to study the boundary stabilizability problem for a general strictly hyperbolic system of conservation laws (1.1) on the domain (1.2) by means of controls acting only at one boundary point, say at the right end of the interval $\mathbb{D}_t$. This corresponds to set $\gamma_0 \equiv 0$ and $\gamma_1(t) = g(\alpha(t))$ in (1.7), for some smooth map $g : \mathbb{R}^p \to \mathbb{R}^p$, and boundary input control $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$. Thus, we will consider the boundary conditions

$$\begin{cases} b_0\big(u(t,\psi_0(t))\big) = 0\,, \\ b_1\big(u(t,\psi_1(t))\big) = g(\alpha(t))\,, \end{cases} \qquad t > 0\,, \tag{1.14}$$

and we shall assume that the maps b_0, b_1, g satisfy the rank conditions (1.8) and

$$\operatorname{rank}\big([Db_0(u)\cdot r_1(u) \mid \cdots \mid Db_0(u)\cdot r_p(u)]\big) = n - p \qquad \forall\, u \in \Omega\,, \tag{1.15}$$

$$\operatorname{rank}\big(Dg(\alpha)\big) = p \qquad \forall \alpha\,. \tag{1.16}$$

Here, because of (1.6), the full rank condition (1.16) guarantees that the total control on the boundary values is available at the right boundary $x = \psi_1(t)$, while thanks to (1.15) one achieves the total control of the boundary values on the left boundary $x = \psi_0(t)$ by means of waves that reach ψ_0 after being generated at ψ_1. Notice that (1.15) in particular implies $p \geq n - p$, i.e. the number of controllable modes (that in this case is the number of characteristic speeds entering the domain from the right boundary ψ_1) must be larger than the number of the uncontrollable ones (the characteristic speeds entering the domain from the left boundary ψ_0 where there is no control). The main result of this paper shows that, under the assumptions (1.8), (1.15), (1.16), the system (1.1) with characteristic speeds satisfying (1.4)-(1.6), and boundary conditions (1.14), can be asymptotically stabilized to any equilibrium states u^*, i.e. to all constant states u^* that satisfy $b_0(u^*) = 0$, starting with an initial condition ϕ with sufficiently small total variation and taking values in a neighborhood of u^*. The study of boundary control problems for conservation laws is motivated by applications to traffic flow models, multicomponent cromatography, electrophoresis, as well as in problems of oil reservoir simulation and gas dynamics (as the one discussed in the above example).

We recall that in the case of hyperbolic systems of conservation laws, because of the nonlinear dependence of the characteristic speeds $\lambda_i(u)$ on the state variable u, classical solutions may develop discontinuities (shock waves) in finite time, no matter of the regularity of the initial and boundary data. Hence, it is natural to investigate the boundary control problems for (1.1) within the context of weak solutions in the sense of distributions. Moreover, for sake of uniqueness, we shall consider weak solutions that are *entropy admissible* in the sense of Lax, i.e. that satisfy an additional admissibility criterion, the classical Lax stability condition [**Da, La**], to single out the physical relevant discontinuities.

DEFINITION 1.1. A function $u \in \mathbb{L}^1_{loc}(\mathbb{D}; \mathbb{R}^n)$ is an *entropy weak solution* of the initial-two-boundaries value problem (1.1), (1.7), (1.9) on $\mathbb{D}$ if the following properties hold:

(1) the map $t \mapsto u(t,\cdot)$ is continuous as a function from $\mathbb{R}_{\geq 0}$ into $\mathbb{L}^1$;
(2) u is a distributional solution to (1.1) on $\mathbb{D}$ in the sense that, for any smooth function ϕ with compact support contained in $\mathbb{D}$, there holds

$$\int_0^\infty \int_{\psi_0(t)}^{\psi_1(t)} \Big[u(t,x)\phi_t(t,x) + F\big(u(t,x)\big)\phi_x(t,x)\Big]\,dx\,dt = 0\,;$$

(3) the initial condition (1.9) is fulfilled;
(4) for all except at most countably many $t \geq 0$ there holds

$$\lim_{x \to \psi_0(t)+} b_0\big(u(t,x)\big) = \gamma_0(t)\,, \qquad \lim_{x \to \psi_1(t)-} b_1\big(u(t,x)\big) = \gamma_1(t)\,; \tag{1.17}$$

(5) at any point of jump discontinuity for u, the left and right limits u^L and u^R, and the speed λ of the jump satisfy the inequalities

$$\lambda_k(u^L) \geq \lambda \geq \lambda_k(u^R)\,, \tag{1.18}$$

for some $k \in \{1,\ldots,n\}$.

We point out that several formulations of the hyperbolic boundary conditions were considered in the literature (see [**Am, JL, AB, Se**]), but they all can be expressed in the form (1.17) in the case of non-characteristic boundaries. For a general account on the basic properties of weak solutions of conservation laws we refer to [**Br2**].

Notice that, for scalar, convex conservation laws, and for Temple systems with genuinely nonlinear characteristic fields, relying on the characterization of the attainable states obtained in [**AM1**], [**AM2**], [**AC**] one can actually show that it is possible to exactly reach a constant state in finite time with an entropy weak solution whose total variation remains small for all times. On the contrary, by the example provided in [**BC**] concerning a particular class of hyperbolic, genuinely nonlinear systems (containing a model studied by Di Perna [**Di**] that describes the isentropic evolution of a polytropic gas), one cannot expect to achieve, in general, such a tipe of exact controllability results. Indeed, for the class of systems considered in [**BC**], there exist initial data which yield a solution that can never be reduced to a constant since it presents a dense set of shock waves, generated by subsequent interactions occuring in the domain $\mathbb{D}$, that remain within $\mathbb{D}$ for all times $t > 0$, no matter which boundary data are assigned. Within the context of weak solutions for general hyperbolic systems it is thus more appropriate to consider instead the problem of asymptotic stabilization near a constant state. A first result in this direction was obtained in [**BC**] where it is shown that, if total control on the boundary values is available at both boundary points, then a general system of conservation laws can asymptotically steer any initial condition ϕ with sufficiently small total variation to all nearby constant state u^*. In this case, since one is assuming that all the components of the solutions entering from the boundaries can be assigned by the controller, the boundary controls can be arranged so that no reflected waves ever enter the domain $\mathbb{D}$ from the boundaries ψ_0, ψ_1, and thus the asymptotic stabilizability problem reduces to prove the existence of an entropy weak solution $u(t, \cdot)$ of (1.1) defined on the domain $\mathbb{D}$ that satisifes the initial condition ϕ and approaches u^* as $t \to \infty$. Clearly, this argument fails when we consider boundary stabilizability problems where the control acts only at one boudary point as in (1.14). In this case the basic strategy to steer the system (1.1) from a given initial state u' at a time t' to a (sufficiently close) equilibrium state u'' within some time t'' consists: firstly, in using the controller acting on the right boundary ψ_1 to generate slow waves that cross transversally the domain $\mathbb{D}$ and reflect on the left boundary ψ_0 producing fast waves that reach, at a time $t^m > t'$, an intermediate state u^m which is connected to u'' solely by waves with negative characteristic speeds; next, in using the controller to generate such slow waves entering from ψ_1 so to reach the desired state u'' at a later time $t'' > t^m$. In effect, since the system is nonlinear, the waves produced by the boundary control ("first generation waves") interact with each other generating new waves ("second generation waves"), and hence the solution u constructed in this way will not attain exactly the terminal state u'' within time t''. However, the complete control of the boundary values along ψ_1 guarantees that all such newly generated waves exit from the domain $\mathbb{D}$ in finite time either reaching directly the right boundary ψ_1 (it is the case of the second generation fronts with positive characteristic speeds), or hitting ψ_1 after being reflected on the left boundary ψ_0 (in the case of second generation fronts with negative characteristic speeds). Therefore, the only wave-fronts that can be present in the solution u at a (sufficiently large) time t'' are those generated by further interactions occuring among the first and second order generation waves, whose total strength is of quadratic order with respect to $|u'' - u'|$, in such a way that $\sup_{x \in \mathbb{D}_{t''}} |u(t'', x) - u''| = \mathcal{O}(1) \cdot |u'' - u'|^2$. Repeating inductively the same procedure on a sequence of time intervals $[t'_k, t''_k]$, $k \geq 1$, we thus construct a boundary control and, correspondingly, an entropy weak solution of (1.1), (1.7), which satisfies the estimate $\sup_{x \in \mathbb{D}_{t''}} |u(t''_k, x) - u''| = \mathcal{O}(1) \cdot |u'' - u'|^{2^k}$ for all k, and hence asymptotically drives the constant state u' to the desired state u''.

We recall that, for systems (1.1) with genuinely nonlinear or linearly degenerate characteristic fields, in the case of initial and boundary data with small total variation the existence of global entropy weak solution of the corresponding mixed problem was established in a series of papers [**Li1, Li2, Go, DG, ST**] using an adaptation of the Glimm scheme, and in [**Am**] developing a front tracking algorithm. More recently, the Lipschitz continuous dependence on the initial and boundary data of entropy weak solutions constructed as limits of front tracking approximations was obtained in [**AC1**] for systems of two equations and in [**DM**] for $n \times n$ systems. All the results quoted so far, with the exception of [**ST**], refer to a mixed problem with a single boundary. An alternative method to establish the well-posedness theory for the mixed problem was purseud in [**AB, Sp**] where the existence and stability of entropy weak solutions was obtained via the study of viscous parabolic approximations $\partial_t u + \partial_x F(u) = \mu\, \partial^2_{xx} u$, as the viscosity coefficient $\mu \to 0$, both in the case of a general $n \times n$ system with a single boundary [**AB**], and in the case of a system of two equations with two boundaries [**Sp**].

Towards the study of the boundary control problem (1.1), (1.14), in the present paper we extend the well-posedness theory for $n \times n$ systems with genuinely nonlinear or linearly degenerate characteristic fields to the case of a domain with two boundaries. Namely, for a closed set $\mathcal{D} \subset \mathbb{L}^1(\mathbb{D}_0; \mathbb{R}^n) \times \mathbb{L}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^p) \times \mathbb{L}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^{n-p})$ of triples of functions with small total variation, we construct by a front tracking algorithm a continuous flow

$$(\phi, \gamma_0, \gamma_1) \;\mapsto\; E_t(\phi, \gamma_0, \gamma_1)\,, \qquad (\phi, \gamma_0, \gamma_1) \in \mathcal{D}\,, \quad t > 0\,, \tag{1.19}$$

of entropy weak solutions of the mixed problem (1.1), (1.7), (1.9) that satisfy the stability estimate

$$\big\| E_t(\phi, \gamma_0, \gamma_1) - E_t(\widehat{\phi}, \widehat{\gamma}_0, \widehat{\gamma}_1) \big\|_{\mathbb{L}^1(\mathbb{D}_t)} \leq L \Big[\|\phi - \widehat{\phi}\|_{\mathbb{L}^1(\mathbb{D}_0)} + \sum_{\iota=0,1} \|\gamma_\iota - \widehat{\gamma}_\iota\|_{\mathbb{L}^1([0,t])} \Big] \quad \forall\, t, \tag{1.20}$$

for some positive constant L. Based on this result, and relying on [**AC2**], one can then derive the uniqueness of solutions to the mixed problem showing that any entropy weak solution of (1.1), (1.7), (1.9) must necessarily coincide with the corresponding map $u(t,x) \doteq E_t(\phi, \gamma_0, \gamma_1)(x)$ induced by (1.19), provided that its total variation does not grow too widly.

The proof of (1.20) is acheived following the same Lyapunov-type approach firstly introduced by Liu and Yang [**LY**], and then developed in [**BLY**], to establish a stability estimate for solutions of the Cauchy problem, which consists in constructing a functional $\Gamma = \Gamma(u, \widehat{u})$, equivalent to the $\mathbb{L}^1$ distance, and almost decreasing along pairs $u, \widehat{u}$ of approximate solutions:

$$\Gamma\big(u(t,\cdot), \widehat{u}(t,\cdot)\big) - \Gamma\big(u(s,\cdot), \widehat{u}(s,\cdot)\big) \leq \mathcal{O}(1) \cdot \varepsilon(t - s) \qquad \forall\, t > s \geq 0\,, \tag{1.21}$$

(ε being a parameter that controls the accuracy of the approximation). The construction of a functional of this kind was recently carried out in [**DM**] for solutions of a mixed problem with a single boundary, and is extended here to the case of an initial-two-boundaries value problem.

The main source of thecnical difficulty determined by the presence of two boundaries, both in providing a front tracking algorithm and in producing a functional that satisfies (1.21), is the possible occurence of repeated reflections of wave-fronts (or of small perturbations measuring the distance between two solutions), bouncing back and forth between the two boundaries ψ_0, ψ_1, that make increase indefinitely their sizes. To overcome this problem, in proving the existence of front tracking solutions defined for all times we shall assume that ψ_0, ψ_1 satisfy a contraction condition quite similar to the one that was considered in [**ST**] to achieve the construction of approximate solutions defined by a Glimm scheme. A stronger assumption will be needed to obtain the estimate (1.21), which is analogous to the condition assumed in [**Le, LT**] to ensure the stability of solutions in presence of two large shocks.

The outline of the paper is the following. Section 2 contains the main assumptions on (1.1) and the boundary data, Section 5 is devoted to the well-posedness theory, first with small BV data, and then for data with "large" total variation, in a sense that will be specified later. Finally, Section 6 illustrates the result regarding the control problem.

2. Preliminaries and statements of the main results

2.1. Notations and basic definitions. In connection with the system (1.1), for every $u \in \Omega$ let $\sigma \mapsto R_i(\sigma)[u]$ denote the *i-th rarefaction curve* through u, i.e. the integral curve of the characteristic eigenvector r_i with starting point u, so that

$$\frac{d}{d\sigma}R_i(\sigma)[u] = r_i\big(R_i(\sigma)[u]\big), \qquad R_i(0)[u] = u,$$

and denote $\sigma \mapsto S_i(\sigma)[u]$ the *i-th Hugoniot curve* through u, which is the the curve of states that satisfy the Rankine-Hugoniot equations

$$F\big(S_i(\sigma)[u]\big) - F(u) = \lambda_i(\sigma)[u]\big[S_i(\sigma)(u) - u\big], \tag{2.1}$$

for some shock speed $\lambda_i(\sigma)[u]$, starting with $S_i(0)[u] = u$. As it is well known [**La, LiY**], the basic building block to construct general solutions of a system of conservation laws is provided by the Riemann problems, i.e. by the initial value problem and by the mixed problem where the data are piecewise constant with a single jump. Since (1.1) has characteristic speeds satisfying either of the assumptions (1.4), (1.5), the solution of a (standard) Riemann problem at a point (τ, ξ), i.e. of a Cauchy problem with initial datum

$$u(\tau, x) = \begin{cases} u^L & \text{if } \; x < \xi, \\ u^R & \text{if } \; x > \xi, \end{cases} \tag{2.2}$$

where $u^L, u^R \in \Omega$, is expressed in terms of the $\mathcal{C}^2$ elementary curves (see [**Br2**, Chapter 5])

$$\Psi_i(\sigma)[u] = \begin{cases} R_i(\sigma)[u] & \text{if } \; \sigma \geq 0, \\ S_i(\sigma)[u] & \text{if } \; \sigma < 0. \end{cases} \tag{2.3}$$

Namely, it consists of $n+1$ constant states $u^0 = u^L, u^1, \dots, u^n = u^R$, and of n elementary waves connecting every pair of adiacent states u^{i-1}, u^i, whose sizes $\sigma_1, \dots, \sigma_n \in \mathbb{R}$ are uniquely determined by

$$u^R = \Psi_n(\sigma_n) \circ \cdots \circ \Psi_1(\sigma_1)[u^L], \tag{2.4}$$

provided that $|u^R - u^L|$ is sufficiently small. Each elementary wave will be either a centered rarefaction wave (when $\sigma_i > 0$) or an admissible shock travelling with speed $\lambda_i(\sigma_i)[u^{i-1}]$ (when $\sigma_i \leq 0$) in the case the i-th characteristic family is genuinely nonlinear, while it is a contact discontinuity travelling with speed $\lambda_i(u^{i-1}) = \lambda_i(u^i)$ in the linearly degenerate case.

Instead, the solution of a (right) boundary Riemann problem at a point $(\tau, \psi_1(\tau)$, i.e. of a mixed problem on the domain $\{(t,x) \mid t > \tau,\, x < \psi_1(t)\}$ with constant initial and boundary data

$$\begin{cases} u(\tau, x) = \overline{u} & x < \psi_1(\tau), \\ b_1\big(u(t, \psi_1(t))\big) = \overline{\gamma} & t > \tau, \end{cases} \tag{2.5}$$

where $\overline{u} \in \Omega, \overline{\gamma} \in \mathbb{R}^p$, according with [**Go, Am**] consists of $p+1$ constant states $u^0 = \overline{u}, u^1, \dots, u^p$, connected by p elementary waves whose sizes $\sigma_1, \dots, \sigma_p$ are uniquely determined by

$$\overline{\gamma} = b_1\Big(\Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(\overline{u})\Big)\,, \tag{2.6}$$

provided that $|\overline{\gamma} - b_1(\overline{u})|$ is sufficiently small. Such a solution coincides with the restriction to the domain $\{(t,x) \mid t > \tau,\ x < \psi_1(t)\}$ of the solution of the (standard) Riemann problem at $(\tau, \psi_1(\tau)$, with initial states $u^L \doteq \overline{u}$, $u^R \doteq \Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(\overline{u})$.

Similarly, the (left) boundary Riemann problem on the domain $\{(t,x) \mid t > \tau,\ x > \psi_0(t)\}$ with constant initial and boundary data

$$\begin{cases} u(\tau, x) = \overline{u} & x > \psi_0(\tau)\,, \\ b_0\big(u(t, \psi_0(t))\big) = \overline{\gamma} & t > \tau\,, \end{cases} \tag{2.7}$$

where $\overline{u} \in \Omega, \overline{\gamma} \in \mathbb{R}^{n-p}$, is solved by $n-p$ elementary waves of sizes $\sigma_{p+1}, \ldots, \sigma_n$ that connect $n-p+1$ constant states $u^p, u^{p+1}, \ldots, u^n = \overline{u}$, and that are uniquely determined by

$$\overline{\gamma} = b_0\Big(\Psi_{p+1}(\sigma_{p+1}) \circ \cdots \circ \Psi_n(\sigma_n)(\overline{u})\Big)\,, \tag{2.8}$$

provided that $|\overline{\gamma} - b_0(\overline{u})|$ is sufficiently small. Such a solution is the restriction to the domain $\{(t,x) \mid t > \tau,\ x > \psi_0(t)\}$ of the solution of the (standard) Riemann problem with initial states $u^L \doteq \Psi_{p+1}(\sigma_{p+1}) \circ \cdots \circ \Psi_n(\sigma_n)(\overline{u})$, $u^R \doteq \overline{u}$.

To establish the existence of more general entropy weak solutions of the hyperbolic system (1.1) on the domain (1.2), we will implement a front tracking algorithm which yields piecewise constant approximate solutions enjoying the properties stated in the following

DEFINITION 2.1. Given $\varepsilon > 0$, an initial condition $\phi \in \mathbb{L}^1(\mathbb{D}_0; \mathbb{R}^n)$, and a pair of boundary data $\gamma_0 \in \mathbb{L}^1_{loc}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n-p})$, $\gamma_1 \in \mathbb{L}^1_{loc}(\mathbb{R}_{\geq 0}; \mathbb{R}^p)$, we say that a continuous map

$$t \mapsto u(t, \cdot) \in \mathbb{L}^1_{loc}(\mathbb{D}_t; \mathbb{R}^n) \qquad t \geq 0\,,$$

is an ε-approximate front tracking solution of the initial-two-boundaries value problem (1.1), (1.7), (1.9) if the followings hold.

(1) As a function of two variables $u = u(t,x)$ is piecewise constant with discontinuities occurring along finitely many straight lines in the domain $\mathbb{D}$. Jumps can be of two types: physical wave-fronts (shocks, contact discontinuities or rarefactions) and non-physical ones, denoted, respectively, as $\mathcal{P}$ and $\mathcal{NP}$. Every interaction involves exactly two incoming fronts.

(2) Along each physical front $x = x_\alpha(t), \alpha \in \mathcal{P}$, the left and right limits $u^L \doteq u(t, x_\alpha(t)-)$, $u^R \doteq u(t, x_\alpha(t)+)$, are related by

$$u^R = \Psi_{k_\alpha}(\sigma_\alpha)\,[u^L]\,, \tag{2.9}$$

for some wave size σ_α and some characteristic family $k_\alpha \in \{1, \ldots, n\}$. In the case the jump (u^L, u^R) is a rarefaction front of a genuinely nonlinear family, one has $0 < \sigma_\alpha \leq \varepsilon$. Moreover, the speed $\dot{x}_\alpha$ of the wave-front satisfies

$$\big|\dot{x}_\alpha - \lambda_{k_\alpha}(\sigma_\alpha)[u^L]\big| \leq \varepsilon \qquad \text{if } (u^-, u^+) \text{ is a shock or a contact discontinuity,}$$

$$\big|\dot{x}_\alpha - \lambda_{k_\alpha}(u^R)\big| \leq \varepsilon \qquad \text{if } (u^-, u^+) \text{ is a rarefaction front.}$$

(3) All non-physical fronts $x = x_\alpha(t), \alpha \in \mathcal{N}$ have the same speed

$$\dot{x}_\alpha \equiv \widehat{\lambda},$$

where $\widehat{\lambda}$ is a fixed constant strictly larger than all characteristic speeds, i.e.

$$\widehat{\lambda} > \lambda_i(u) \qquad \forall\, i = 1, \ldots, n, \quad \forall\, u \in \Omega. \tag{2.10}$$

Moreover, the total strength of all non-physical fronts in $u(\cdot, t)$ is uniformly small, i.e.

$$\sum_{\alpha \in \mathcal{NP}} \big|u(t, x_\alpha(t)+) - u(t, x_\alpha(t)-)\big| \leq \varepsilon, \qquad \forall t \geq 0. \tag{2.11}$$

(4) The initial and boundary values of u fulfill approximatively the initial and boundary conditions, i.e.

$$\|u(0, \cdot) - \phi\|_{\mathbb{L}^1} \leq \varepsilon, \tag{2.12}$$

$$\|b_\iota\big(u(\cdot, \psi_\iota(\cdot))\big) - \gamma_\iota\|_{\mathbb{L}^\infty} \leq \varepsilon \qquad \iota = 0, 1. \tag{2.13}$$

Throughout the paper, with a slight abuse of notation, we shall often call σ a wave of size σ, and, if $u^R = \Psi_k(\sigma)[u^L]$, we will say that (u^L, u^R) is a (physical) wave of size σ of the k-th characteristic family. We shall define as $|u^R - u^L|$ the size (and strength) of a non-physical front joining two states u^L, u^R.

Front tracking solutions of the mixed problem (1.1), (1.7), (1.9) are constructed by piecing together several (approximate) solutions of Riemann problems that either are generated by the jumps in some piecewise constant approximation of the initial and boundary data, or arise at the interaction points between wave-fronts. This construction can be carried out for all times provided that the corresponding Riemann data remain close to each other, which is certainly guaranteed as long as the total variation of the approximate solution remains uniformly small. To this purpose, in presence of two boundaries we need to impose a contraction condition on the boundary maps ψ_0, ψ_1, so to prevent that repeated bouncings back and forth between ψ_0 and ψ_1 of wave-fronts could indefinitively increase their sizes.

BV stability condition.

Consider the $(n-p) \times p$ matrix

$$M_0(u) \doteq \Big[m^0_{jk}(u)\Big]_{\substack{j=1,\ldots,n-p \\ k=1,\ldots,p}}, \tag{2.14}$$

and the $p \times (n-p)$ matrix

$$M_1(u) \doteq \Big[m^1_{jk}(u)\Big]_{\substack{j=1,\ldots,p \\ k=1,\ldots,n-p}}, \tag{2.15}$$

defined, for $u \in \Omega$, by

$$m^0_{jk}(u) \doteq \left|\Big[Db_0(u)\cdot r_{p+1}(u)\,\Big|\cdots\Big|\,Db_0(u)\cdot r_n(u)\Big]^{-1}_j Db_0(u)\cdot r_k(u)\right|, \tag{2.16}$$

$$m^1_{jk}(u) \doteq \left|\Big[Db_1(u)\cdot r_1(u)\,\Big|\cdots\Big|\,Db_1(u)\cdot r_p(u)\Big]^{-1}_j Db_1(u)\cdot r_k(u)\right|,$$