Progress in Chaotic Dynamics

Essays in honor of Joseph Ford's 60th birthday

Editors:

Hermann Flaschka
Boris Chirikov

NORTH-HOLLAND

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Hermann Flaschka

Mathematics Department
University of Arizona
Tucson, AZ 85721, USA

Boris Chirikov

Institute of Nuclear Physics
630090 Novosibirsk, USSR

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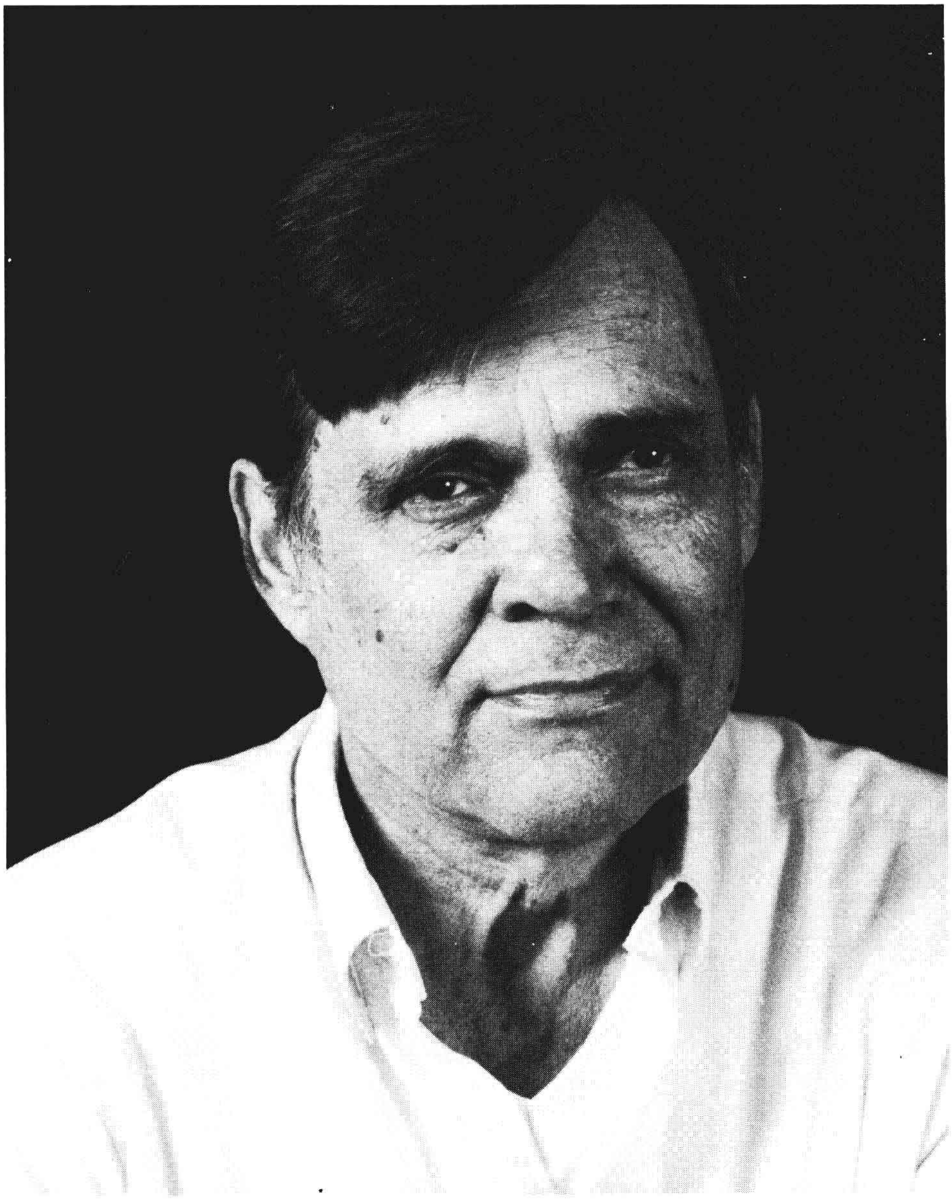

Professor Joseph Ford

Photograph: Vincent Mallette, Georgia Institute of Technology

PREFACE

It is a great pleasure for us to dedicate this special issue of Physica D to Joseph Ford on the occasion of his sixtieth birthday. It seems particularly appropriate to use the pages of this journal, which Joe helped found eight years ago, to express our appreciation for his work.

This is not merely a recognition of Joe's contributions to the field of nonlinear dynamics. It is a tribute to a colleague whose scientific life has been dedicated to forming, developing, and guiding our scientific community. It is also a present to a friend whose generosity and kindness we have admired, and whose sense of humor we have, so many times, enjoyed.

Joe Ford came to the new field of chaotic dynamics after having been diverted from 'conventional' statistical mechanics by the famous Fermi–Pasta–Ulam problem. As early as 1963, he discovered a novel phenomenon, a transition from ordered motion to stochasticity. In those pioneering days, Joe sensed, more than anyone else, the profundity and significance of the new dynamics. As related ideas emerged from diverse fields, he served as catalyst and focal point for the new developments.

A prolific period followed. We just recall his empirical criterion for chaos, based on local instability of motion, his computer proof of the integrability of the Toda lattice, which stimulated intensive investigations of integrable equations, and his study of chaos in systems that can be arbitrarily close to linear oscillators.

As the literature of dynamics began to grow, Joe conceived the Nonlinear Science Abstracts. He collected, organized, typed, copied, and distributed volumes of abstracts of dynamics papers, and thereby made new theory and new applications available to an interdisciplinary audience. We thank Joe for all the days spent at the typewriter, and for all the weekends with the photocopy machine.

As new researchers rushed into nonlinear dynamics, Joe retreated into lonelier domains, to explore deeper consequences of deterministic chaos. He realized that the concept of chaos transcends the domains of mappings and differential equations, and drew attention to the notion of algorithmic complexity as a means of defining and assessing the limits of our ability to deal with chaotic systems.

With this tool, he began daring investigations in a field which would come to be called Quantum Chaos. He perceived that such chaos does not exist at all, and unlike others who wished to explore the Quantum maze, he challenged quantum mechanics itself.

A very good start for the first half-century or so, Joe. Forward and good luck!

Giulio Casati
Boris Chirikov
Hermann Flaschka
Franco Vivaldi

who are responsible for this Foreword

and all the people who filled the
rest of this volume.

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THE ONSET AND SPATIAL DEVELOPMENT OF TURBULENCE IN FLOW SYSTEMS

I.S. ARANSON, A.V. GAPONOV-GREKHOV and M.I. RABINOVICH
Institute of Applied Physics, Academy of Sciences of the USSR, Gorky, USSR

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The self-generation and development of spatio-temporal chaos – turbulence – in flow systems, in particular in shear hydrodynamical flows, are analysed experimentally and theoretically. We construct chain models of such systems and use them for the investigation of the spatial bifurcations preceding the appearance of chaos (period doubling downstream, transition via quasiperiodicity, and so on). We give the renormalization group description of the complication of dynamics along the chain and study the "order–chaos" transition waves. The concepts of convective and absolute development of chaos in space are introduced for flow systems. The prospects of future investigations are discussed.

1. Introduction

1.1. There hardly exists a phenomenon that could compete with turbulence in its wide interest, accompanied by impetuous discussions and contradictory statements of physicists, mathematicians and engineers. This is connected with the exceptional complexity of the problem as well as with significant differences in the ways it is understood by engineers, mathematicians and physicists. In terms of an applied problem, turbulence reduces to the formulation of some effective equations (which are much simpler than the Navier–Stokes equations) for the calculation of drag, heat and mass transfer, and other characteristics of turbulent flows. Such equations can be based on intuitive knowledge as well as on various semiempirical hypotheses. The main goal of these equations is to make predictions for real experimental situations; remarkable results have been obtained (see, for example, [1]). However, from the viewpoint of a physicist such an approach to turbulence leaves aside some fundamental questions. For example: Why does a laminar (ordered) flow become disordered? How does this disorder effect the mean characteristics of the flow? It is essential to understand whether the mixing of elements in the fluid results from the uncontrolled pulsations of the flow around the body or is due to the instabilities inherent in the flow. Noise and fluctuations play an important role, no doubt, but only as a "triggering mechanism" to initiate instability.

These are questions of principle for any type of turbulence: hydrodynamical, plasma, sleak, and so on. In other words, we deal with a fundamental problem – how a nonlinear field of arbitrary origin transforms into a disordered, random motion and how this motion is to be described.

The evolution of the viewpoint on the origin of randomness in the theory of turbulence and that in statistical physics have much in common. The concepts formulated by Osborn Reynolds for small perturbations growing as a result of linear instability could not explain the transition to turbulence in simple flows such as Poiseuille flow and Couette flow between planes. According to the linear theory, there exists no critical Reynolds number within the framework of the Navier–Stokes equations for a laminar flow to become unstable in a tubular current. However, all experimenters know only too well that such a flow becomes turbulent when Re ~ 5000. That, and other "failures" of the linear theory, called into question the validity of the deterministic

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Navier–Stokes equations for the description of turbulent flows. Many research scientists, for example, Kármán and Taylor, believed that similarly to the motion of gases, turbulence can be understood and interpreted only using a statistical approach. Their standpoint was supported by the fact that for sufficiently large Reynolds numbers the number of the degrees of freedom involved in the motion is so large that only a mean field description of the flow is possible.

The first attempts to explain a disordered, chaotic "water flow" in terms of pure dynamics were undertaken by Landau (1944) [2] and Hopf (1948) [3]. Their models, based on the same principle, implemented the concept of successive complication of the flow due to a developing hierarchy of instabilities with incommensurate time scales. Nonlinear stabilization of these instabilities results in a flow where the degree of disorder in the velocity field grows with the increase of the number of perturbations with incommensurate scales that are involved in the formation of the flow. In the phase space of such a flow, a trivial equilibrium state produces a limit cycle (the image of a single-frequency periodic flow), then the cycle becomes unstable and the bifurcation generates an attractor in the form of a dense winding on a two-dimensional torus. After the n-th bifurcation there appears an attractor in the form of an ergodic winding on a n-dimensional torus, and so on.†

The Landau–Hopf hypothesis played an important role in the development of the concept of the dynamical origin of turbulence, even though the models proposed four decades ago proved to be inadequate. The point is that the attractor in the form of a dense winding on a multidimensional torus, which is the image of turbulence in terms of such models, is structurally unstable, i.e., it transforms into a limit cycle or a strange attractor even with a small variation of the parameters of the system. This means that such a disordered flow, or to be more exact, a complex flow with a large number of incommensurate frequencies in the Fourier spectrum, cannot be realized in general.

Nevertheless, the supposition that the development of turbulence depends on successive excitation (by increase of the Reynolds number) of new degrees of freedom in the flow proved to be correct.

†The Landau–Hopf scenario was recently observed in a chain of unidirectionally coupled rotators [4].

1.2. When we speak about the relation between dynamical chaos and turbulence in a real water flow there arises the question whether the dynamical theory can describe the variation of the characteristics of turbulence in space, in particular, whether the dynamical theory can explain the phenomenon of spatial development of turbulence downstream in shear hydrodynamical flows.

Note that the dynamical theory can be used for the description of the onset of turbulence in flows in cells; for example, thermoconvection in a cell, or Couette–Taylor flow between rotating cylinders. This is confirmed by a good agreement between the experimentally observed scenarios and the theory, as well as by direct measurements of the parameters and the characteristics of a strange attractor, primarily its dimension past the critical point. For such flows the mean characteristics of turbulence (at least, for moderate supercriticalities, i.e., for Rayleigh and Taylor numbers) can be considered to be independent of coordinates. The predictions of the dynamical theory can be verified in this case by experimental data obtained from the measurements of the time series, for example, the velocity field at one point of the flow. Remarkable results on this problem were obtained by Brandstäter and Berge and their collaborators [5, 6].

For shear flows, the problem is more complicated, even in the way it is posed. The main issues are the following:

(1) Does spatial anisotropy of the flow guarantee the anisotropy of the mean characteristics of a turbulent regime?

(2) How is dynamical chaos generated downstream? Are the spatial scenarios of the transition to chaos similar to the scenarios in simple systems with a varying control parameter?
(3) What is the asymptotic behavior of the chaotic regime developing indefinitely far downstream? These and related problems are considered in our paper.

In section 2, we discuss the results of experiments on shear flows (flow jets) and a periodically excited boundary layer on a plate. The correlation dimension is determined as a function of downstream coordinates.

In section 3, we construct dynamical models of flow systems in the form of directionally coupled chains of nonlinear oscillators or maps. The restructuring of the flow dynamics along the chain – spatial bifurcation – is analysed and a renormalization group description of such bifurcations is given.

In section 4, we study solutions describing the moving fronts of the "order–chaos" and "chaos–chaos" transitions in chains with different dynamics of individual elements. The problem of the finite number of spatial bifurcations preceding the birth of chaos along the chain is considered. The critical indices which define the propagation velocity of the moving transition fronts are determined. We believe that the effect of saturation of dimension along the chain is related to stochastic synchronization.

In section 5, we discuss the prospects of further investigation of dynamical chaos in flow systems encountered in Nature and induced by theoretical studies.

Details of calculations are presented in appendices A and B.

2. Experimental results

First of all we shall present the results of two experiments on shear flows: a jet flow with acoustic feedback and a boundary layer on a plate. One task was to be solved in both experiments: how the dimension of a chaotic regime (turbulence) depends on spatial coordinates. The treatment was carried out as follows. Long-time dependence of the velocity field was measured at several points downstream (about 10–20 points in space). Based on those measurements, we calculated the correlation integrals with the downstream coordinate being a parameter. The data obtained were used to plot the dimension versus the coordinate along the flow [7, 8].

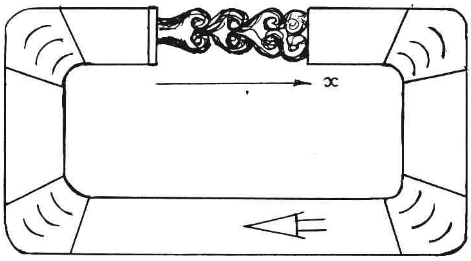

Fig. 1. Wind tunnel with open operating part.

2.1. The jet flow was formed in an open part of a closed wind tunnel (fig. 1). Acoustic feedback was provided through the return channel of the wind tunnel. The hydrodynamical perturbations were transformed into acoustical ones at the edge of the exit cone. The inverse transformation took place at the nozzle output. The velocity pulsations were measured by a hot-wire anemometer, and the pressure pulsations by a condenser microphone. The nature of the velocity pulsations propagating downstream changed but slightly. The correlation dimension remained constant along the flow and changed only with the variation of the control parameter – the flow velocity V_0 (fig. 2).

Thus, due to the feedback, the development of turbulence (dynamical chaos) in the shear flow considered took place simultaneously throughout space. By analogy with a known absolute instability, such a regime may be called absolute (homogeneous) development of chaos.

It seems natural that for a rather weak feedback which depends, for example, only on spontaneous diffusion of the velocity field, a qualitatively new regime is to be realized, the regime where the dimension of the dynamical chaos increases along the flow similarly to convective gain. To verify this hypothesis we also studied the evolution of dynamics along the flow in a boundary layer on a plate.

2.2. Our investigations were carried out in a low-turbulence ($\sim 0.002\%$) wind tunnel. The flow velocity was taken to be 9.18 m/s. The boundary layer was formed on a smooth 1.5 m long plate with a rounded leading edge. A vibrating ribbon was used to introduce periodic, nearly sinusoidal, perturbations exciting quasi-two-dimensional Tollmien–Schlichting waves in the boundary layer.

Let us now introduce the correlation dimension of the flow based on the following considerations. Assume that the time dependence of the flow velocity $V_x(t)$ measured by the hot-wire anemometer at the point x downstream is produced by the dynamical system $G_x(y)$ whose motion in the M_x-dimensional phase space is described by the trajectory $y(t)=\{V(t),\ V(t+\tau),\ V(t+2\tau),\dots,V(t+(M-1)\tau)\}$. In order to calculate the dimension we use the following correlation integral [9]:

$$C_x(r)=\frac{1}{N_0^2}\sum_i\sum_j H(r-\|y_i-y_j\|)=\frac{\langle N_r^i(x)\rangle}{N_0}=\frac{N_r(x)}{N_0}, \tag{1}$$

where $H(r)$ is the Heaviside function, $y_i(x)$ is a point in the M_x-dimensional phase space, $N_r^i(x)$ is the number of points in the r-neighborhood of the ith point, $N_r(x)$ is their mean number in the spheres with a radius r, and N_0 is the total number of points. The quantization interval in the time τ is chosen in accordance with the sampling theorem $\tau\sim 1/2F_{max}$, where F_{max} is the upper boundary of the spectrum of $V(t)$. We took 10^5 samples for each time dependence of velocity.

The correlation integral (1) in the finite intervals Δr corresponding to different amplitude scales of $V(t)$ is approximated by the power function $C_x(r)\sim r^{\nu(x)}$ and the dimension is estimated directly from

$$\nu(x)=\frac{\ln C_x(r+\Delta r)-\ln C_x(r)}{\ln\Delta r}.$$

The measured magnitude of the dimension depends on the range of the velocity pulsations of interest. Their variations, which are much smaller than Δr, do not contribute to the calculated values of the dimension $\nu(x)$. Such a differentiation in the contribution of various scales of $V(t)$ enables

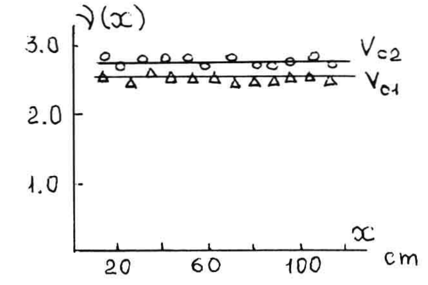

Fig. 2. Correlation dimension versus x for different values of V_0: $V_{01}=10.2$ m/s and $V_{02}=24.2$ m/s.

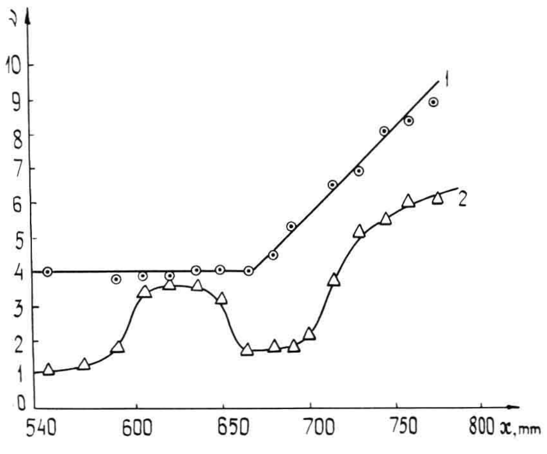

Fig. 3. Correlation dimension versus longitudinal coordinate: (a) $D_0=4$ and (b) $D_0=1$.

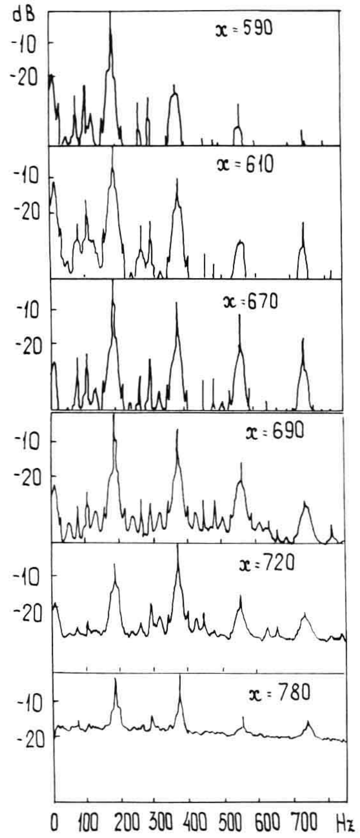

Fig. 4. Power spectra in the boundary layer versus the x-coordinate.

one to estimate the number of modes from $\nu(x)$ and even to identify these modes, provided that additional information is available. This is very important for the construction of a dynamical model for the development of the flow downstream [8].

The measured dimension of the time dependence of velocity is plotted in fig. 3 versus the longitudinal coordinate for two intervals Δr differing by 10 db. Based on the analysis of these dependences and on the spectral data (fig. 4) the time dependence of velocity in the initial portion of the plate $x = 590$ mm (even with small variations in the time dependence of velocity taken into account) can be presented as a result of the excitation of four modes, that is, the modes generated by the ribbon at the frequency of the external force ($\sim$ 87 Hz), at the parasitic frequency ($\sim$ 50 Hz), at the frequency of natural oscillations of the ribbon in the flow ($\sim$ 60 Hz), and at the frequency of the mode excited by the vibrations of the plate ($\sim$ 3 Hz). These modes are, in fact, independent. Large vibrations of the velocity are connected only with one mode excited at the control frequency.

In the region $590 < x < 665$ mm all four modes ($\nu = 4$) should be taken into account even for a rough approximation of $G_x(y)$. However, the dimension for the same approximation of the time dependence of velocity diminishes to two further downstream ($\sim$ 665 mm) in the region where "pins" characteristic of a catastrophic (Klebanoff) transition to turbulence are formed on the oscillograms. The measurements of the cross-correlation of the signal fed to the ribbon and the signal from the hot-wire anemometer show that the Tollmien–Schlichting waves at the control frequency $\sim$ 87 Hz and its phased-in harmonics make the basic contribution to the formation of the pins.

It is well known that strong undershoots on velocity oscillograms (pins) are indicative of the local deceleration of the fluid and the formation of a bending point in the velocity profile, which results in the appearance of secondary instability. The point along the flow ($x = 665$ mm), where new modes are excited as a result of the evolution of this instability, is determined by qualitative variations in the dependence $\nu(x)$. Further downstream ($x = 700$ mm), these modes dominate the formation of pulsations with large amplitudes which become strongly chaotic. The visual observations of the flow show that for $x = 700$ mm the two-dimensional structures are destroyed and three-dimensional structures are formed in the boundary layer.

Thus, the evolution of turbulence in the boundary layer associated with the destruction of two-dimensional structures and the growth of new excitations is a dynamical process whose dimension increases downstream.

3. Models and theory

3.1. Let us assume for simplicity that the cross-section of a shear flow has a fixed structure. Thus we have a one-dimensional problem which is characterized only by one spatial coordinate x increasing along the flow. Let the spatial (along x) spectrum of the field distribution be limited at any instant of time. This is characteristic of any real flow. Then we can construct a dynamical model flow and consider the velocity field only at the following points: $x = x_0 + j\Delta x$, $j = 1, 2, \ldots$. Here Δx is determined as $\sim 1/K_x$ where K_x is the limiting spatial "frequency" of the spatial spectrum. Here we have a complete analog with the sampling theorem. The resulting model has the following form [10–12]:

$$\frac{\mathrm{d}\boldsymbol{u}_j}{\mathrm{d}t} = \Phi(\boldsymbol{u}_j, \lambda) + \chi(\boldsymbol{u}_{j+1} + \boldsymbol{u}_{j-1} - 2\boldsymbol{u}_j) - \gamma(\boldsymbol{u}_j - \boldsymbol{u}_{j-1}) \equiv \hat{F}[\boldsymbol{u}_j]. \tag{2}$$

Here the equation $\mathrm{d}\boldsymbol{u}_j/\mathrm{d}t = \Phi(\boldsymbol{u}_j, \lambda)$ describes the dynamics of the jth element for which λ is the control parameter, γ is responsible for nonmutual coupling between the elements determined by the flow, and χ stands for diffusion coefficients; the operator $\hat{F}$ takes into account the shear along j. The natural boundary conditions for (2) are either $\boldsymbol{u}_{j=0} = 0$ or $\boldsymbol{u}_{j=0} = \boldsymbol{f}(t)$, where $\boldsymbol{f}(t)$ is a given function of time t.

Bearing in mind that the dynamical behavior of the individual elements can be described by maps, we shall use at $\chi = 0$, in addition to (2), the system

$$U_j(n+1) = f_0(U_j(n), \lambda) + \gamma_1(U_j(n) - U_{j-1}(n)) + \gamma_2(f_0(U_j(n), \lambda) - f_0(U_{j-1}(n), \lambda)) \tag{3}$$

under the boundary conditions $U_{j=0}(n) = \text{const.}$ Here γ_1 is responsible for the "inertial" coupling and γ_2 is responsible for the "dissipative" coupling.

It is important that for some types of shear flows the spatial discreteness of model (2) is naturally related to the flow structure which is formed as a result of developing initial instabilities. For example [13], a boundary layer on a rotating cone or a cylinder is a chain of vortices thread on the body of rotation, whose dynamics is described by the variable $\boldsymbol{u}_j(t)$. For instance, $\boldsymbol{u}_j(t)$ may be a complex amplitude of the waves excited on the jth vortex.

Model (2) permits us to prove the self-generation of turbulence along the flow and to study the spatial scenarios of the appearance of chaos, that is, the restructuring of the flow along j preceding the turbulence.

Neglecting the diffusion, $\chi = 0$, the problem on the spatial birth of chaos reduces strictly to the problem on the birth of a strange attractor on a certain element of the chain.

For $\chi = 0$, a stationary regime spatially inhomogeneous along j has the following solution:

$$\boldsymbol{u}_j^0 = (\Phi(\boldsymbol{u}_j^0, \lambda) + \gamma \boldsymbol{u}_j^0)/\gamma. \tag{4}$$

The evolution of perturbations $\boldsymbol{\xi}_j(t)$ on the background $\boldsymbol{u}_j^0$ is determined by the system

$$\frac{\mathrm{d}\boldsymbol{\xi}_j}{\mathrm{d}t} = [\Phi_u'(\boldsymbol{u}_j^0, \lambda) + \gamma]\boldsymbol{\xi}_j - \gamma\boldsymbol{\xi}_{j-1}. \tag{5}$$

This system has a block-triangular matrix, therefore the Lyapunov characteristic exponents (5) coincide with the indices of the elementary systems, $\mathrm{d}\boldsymbol{\xi}_j/\mathrm{d}t = [\Phi_u'(\boldsymbol{u}_j^0, \lambda) + \gamma]\boldsymbol{\xi}_j$. If $\boldsymbol{u}_j^0$ and λ enter into the function $\hat{F}[\boldsymbol{u}_j^0]$ similarly, then for small γ this coincidence means that the spatial bifurcations in the chain are similar to those in the point element. The value of U_j^0 varied along j is the control parameter in the spatial problem.

Specifying the form of the function $\Phi(\boldsymbol{u}_j, \lambda)$ one can describe various spatial scenarios of the appearance of chaos along the flow. In particular, if a single element demonstrates a period doubling sequence with the variation of λ and the resulting birth of chaos, for example when $f_0(U_j, \lambda)$ in (3) has a square maximum, then a similar pattern will be observed in space (see fig. 5 (a, b and c)) [14, 15]. The only difference here is that the number of period doublings preceding chaos is always finite (see the text below). The number of spatial bifurcations in the transition to chaos via the destruction of a quasiperiodic regime and in other types of the transition to chaos will also be finite [11]. This transition is described, in particular, by the equation

$$\dot{a}_j = \varepsilon a_j - (1 - \mathrm{i}\beta)|a_j|^2 a_j + \gamma(a_j - a_{j-1}),$$
$$a_{j=0} = A_0 \mathrm{e}^{\mathrm{i}\varepsilon\omega t}, \quad 0 < \varepsilon \ll 1, \quad \beta = \text{const}. \tag{6}$$

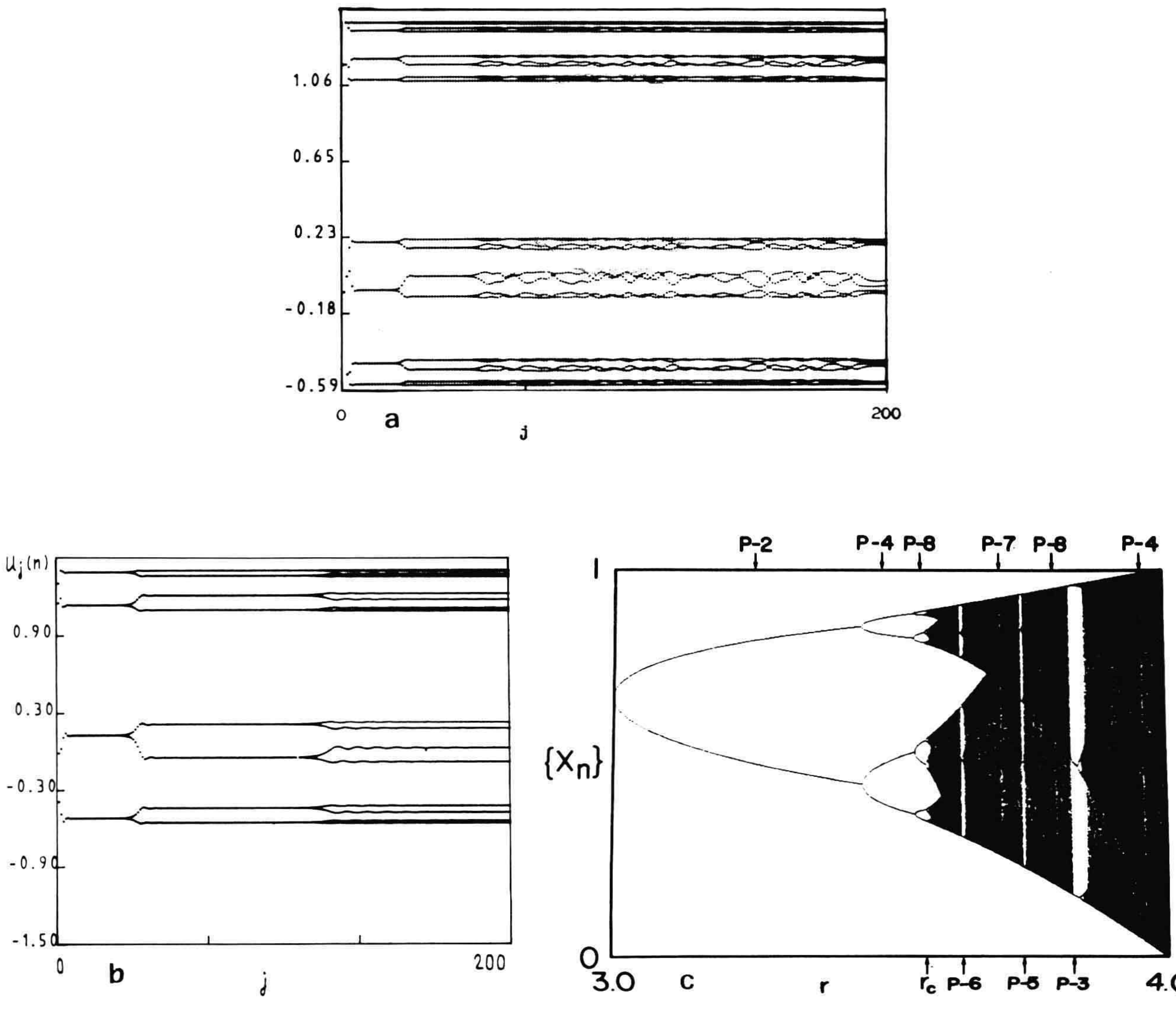

Fig. 5. Period doubling bifurcations: spatial bifurcations in the chain $U_j(n+1) = f(U_j(n)) + \gamma(f(U_{j-1}(n)) - f(U_j(n)))$; $f = \lambda - U^2$ with the following parameters: (a) $\Lambda = 0.001$, $\gamma = 0.01$; (b) $\Lambda = 0.001$; $\gamma = 0.27$; (c) period doubling in a point system, $x_{n+1} = rx_n(1 - x_n)$.

3.2. Assuming $f_0 = \lambda - U_j^2(n)$ we put eq. (3) into the form

$$U_j(n+1) = F_0\big[U_j(n)\big] = f_0\big(U_j(n)\big) + \gamma\varphi_0\big(U_j(n), U_{j-1}(n)\big). \tag{7}$$

Assume that the coupling is weak: $|\gamma| \ll 1$ and $\varphi_0(U, U) = \varphi_0(0, 0) = 0$. The weak coupling ensures smooth dependence of the stationary regime on the j-coordinate in the interval between two spatial bifurcations because the elements are identical (fig. 5). Since period doubling is a sign of critical behavior, our interest is the development of chaos in the chain from the initial 2^N-periodic regime. This chaos development through the sequence of period doublings takes place when $\Lambda = \lambda - \lambda_c > 0$ in all elements,† while $\lambda(1-\gamma) < \lambda_c$, in the first element (assume for definiteness $U_0(n) \equiv 0$). A stable 2^N-periodic regime occurs at the beginning of the chain and then transforms into a chaotic regime with increasing j. Fig. 5(a) demonstrates the spatial development of chaos through the period doubling.†† Fig. 5(a) is similar to the bifurcation diagram for the map $U(n+1) = \lambda - U^2(n)$ by the parameter λ (fig. 5(b)) which determines the dynamical behavior of the point element. This means that the spatial development of chaos along j can be described approximately considering one element with the supercriticality parameter $\Lambda(j)$.

We now construct a renormalization group (RG) equation by means of the scaling transformation (period doubling transformation) of (7). We express the variables through two units of discrete time $U_j(n+2)$ through $U_j(n)$ and make a substitution $U_j \to U_j/a_S$. Up to $\mathcal{O}(\gamma^2)$ this yields

$$\begin{aligned} U_j(n+2) &\approx F_1\big(U_j(n), U_{j-1}(n)\big) \\ &= f_1\big(U_j(n)\big) + \gamma\varphi_1\big(U_j(n), U_{j-1}(n)\big), \\ f_1(U_j) &= a_S f_0 f_0(U_j/a_S), \\ \varphi_1(U_j, U_{j-1}) &= a_S\big(f_0' f_0(U_j/a_S)\varphi_1(U_j/a_S, U_{j-1}/a_S) \\ &\quad + \varphi_0\big(f_0(U_j/a_S), f_0(U_{j-1}/a_S)\big)\big). \end{aligned} \tag{8}$$

Using the procedure N times we obtain the RG equation

$$U_j(n+2^N) = F_N\big(U_j(n), U_{j-1}(n)\big) = f_N\big(U_j(n)\big) + \gamma\varphi_N\big(U_j(n), U_{j-1}(n)\big),$$

which comprises two equations

$$\begin{aligned} f_{N+1}\big(U_j(n)\big) &= a_S f_N f_N\big(U_j(n)/a_S\big), \\ \varphi_{N+1}(U_j, U_{j-1}) &= a_S\big(f_N' f_N(U_j/a_S)\varphi_N(U_j/a_S, U_{j-1}/a_S) \\ &\quad + \varphi_N\big(f_N(U_j/a_S), f_N(U_{j-1}/a_S)\big)\big). \end{aligned} \tag{9}$$

These equations are a particular case of the universal operator RG equation

$$F_{N+1}[U(r)] = S^{-1}F_N F_N S[U(r)], \tag{10}$$

where not only the functions (operators) determining the dynamical behavior of each individual element but also the spatial coordinates undergo scale transformation in the transition to the map through two units of time.† We introduce the notation $S = S_1 S_2$; $S_1[U(r)] = U(r)/a_S$, $S_2[U(r)] = U(r/b_S)$; b_S into the spatial scaling factor. In general, the variable $U(r)$ can be a vector or a matrix. Since the j-coordinate is discrete, the spatial scale in eq. (9) is changed by the renormalization of the coupling γ. Below it will be shown that

†The value of the parameter $\lambda = \lambda_c$ is the critical point in the transition to chaos in each individual element.

††It can be shown (see appendix A) that these spatial bifurcations can be described approximately in terms of the map $\Lambda(j+1) = \psi(\Lambda(j))$.

†Eq. (10) for the transition to chaos in a diffusive medium composed of the elements described by the Feigenbaum map was first investigated in [17].

in the long-wave limit, eq. (9) yields the same scaling laws as eq. (10).

In this case the RG method aims at finding the critical indices and scale factors determining the similarity of spatial bifurcations for a given type of critical behavior in the vicinity of the critical point. However, the solution of the RG equation can be found only at the critical point

$$G[U(r)] = S^{-1}GGS[U(r)], \tag{11}$$

where the operator G is invariant with respect to the RG action. Moreover, the scale constants can be found from the solutions of the RG equation linearized near the fixed point G. In general, this is insufficient, but the problem can be solved with some restrictions on the action of the operator F_N. For this we must calculate the "preceding", increasingly distant iterations of the F_N action from the "subsequent" iterations adjacent to the fixed point of RG and to predict the behavior of the system depending on the parameter value in an expanding neighborhood of the critical point.

Such a calculation is possible if the fixed point of the RG equation has expanding, i.e., unstable directions. Only along these directions does the action of the F_N operator near the critical point determines the behavior of the system far away. We emphasize that the set of unstable directions in the functional space with a given F_N operator cannot be continuous. Otherwise the critical behavior ceases to be universal, since the infinitesimal changes of the initial operator F_0 will lead to absolutely different scaling laws. Thus, the fixed point G of the RG equation describing the universal behavior at the critical point must be of a saddle type. To determine the scaling laws, we need to know the perturbations of the F_N operator which are defined by eigenvalues of the linearized problem with modulus larger than unity. Below, we shall refer to them as *relevant eigenvalues*. Actually, they are the desired constants determining the scale similarity when the transformations are given by the perturbation operators corresponding to these factors.

We now apply these ideas to the RG equation (9). This equation has a fixed point

$$G = \begin{Bmatrix} f_N \\ \varphi_N \end{Bmatrix} = \begin{Bmatrix} g \\ 0 \end{Bmatrix}. \tag{12}$$

Here the g function coincides with the universal Feigenbaum function (period doubling) [16] which is the solution of the functional equation

$$g(U) = a_S gg(U/a_S), \quad a_S = -2.5029\dots. \tag{13}$$

Let us investigate the RG equation (9) in the vicinity of the fixed point (12). The substitution of $F_N(U_j, U_{j-1})$ in the form

$$F_N(U_j, U_{j-1}) = g(U_j) + \varepsilon\left(h_N(U_j) + \gamma\tilde{\varphi}(U_j, U_{j-1})\right) \tag{14}$$

for the h_N and $\tilde{\varphi}_N$ functions in the first approximation in ε yields

$$h_{N+1}(U_j) = a_S\left(g'_U g(U_j/a_S) h_N(U_j/a_S) + h_N\left(g(U_j/a_S)\right)\right), \tag{15}$$

$$\varphi_{N+1}(U_j, U_{j-1}) = a_S\left(g'_U g(U_j/a_S)\tilde{\varphi}_N(U_j/a_S, U_{j-1}/a_S) + \tilde{\varphi}_N\left(g(U_j/a_S), g(U_{j-1}/a_S)\right)\right). \tag{16}$$

Eq. (15) is the linearized Feigenbaum equation at the fixed point $g(U)$ which has therefore a single relevant eigenvalue $\delta = 4.669$ corresponding to the eigenfunction $h_0(U) = 1 + \mathcal{O}(U^2)$. Eq. (16) coincides with the RG equation describing the transition to chaos in the system of two coupled parabolic maps [18]. According to [18] this equation has two relevant eigenvalues: $\nu_1 = a_S = -2.5029\dots$ and $\nu_2 = 2$. The eigenvalue corresponds to the eigenfunction $\tilde{\varphi}_1(U_j, U_{j-1}) = (U_j - U_{j-1})\Phi(U_j, U_{j-1})$, $\Phi(U_j, U_{j-1}) \sim 1$. This type of perturbation can be interpreted as an introduction of inertial coupling between the point elements. The eigenvalue ν_2 corresponds to the