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QUANTUM STOCHASTIC PROCESSES AND NONCOMMUTATIVE GEOMETRY

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Preface

On the one hand, in almost all the scientific areas, from physical to social sciences, biology to economics, from meteorology to pattern recognition in remote sensing, the theory of classical probability plays a major role and on the other much of our knowledge about the physical world at least is based on the quantum theory [12]. In a way, quantum theory itself is a new kind of theory of probability (in the language of von Neumann and Birkhoff) (see for example [106]) which contains the classical model, and therefore it is natural to extend the other areas of classical probability theory, in particular the theory of Markov processes and stochastic calculus to this quantum model.

There are more than one possible ways (see for example [127]) to construct the above-mentioned extension and in this book we have chosen the one closest to the classical model in spirit, namely that which contains the classical theory as a submodel. This requirement has ruled out any discussion of areas such as free and monotone-probability models. Once we accept this quantum probabilistic model, the 'grand design' that engages us is the 'canonical construction of a $*$-homomorphic flow (satisfying a suitable differential equation) on a given algebra of observables such that the expectation semigroup is precisely the given contractive semigroup of completely positive maps on the said algebra'.

This problem of 'dilation' is here solved completely for the case when the semigroup has a bounded generator, and also for the more general case (of an unbounded generator) with certain additional conditions such as symmetry and/or covariance with respect to a Lie group action. However, a certain amount of space has to be devoted to develop the needed techniques and structures, and the reader is expected to be well equipped with the basics of functional analysis, theory of Hilbert spaces and of operators in them and of probability theory in order to master these.

A beginner with the above-mentioned background may read Chapters 1 to 6 at first and may leave the rest for a second reading. In some places, mathematical assertions have been made without proof wherever we felt that the proof is essentially similar to a detailed proof of an earlier statement or when the verification of the same can be left as an exercise.

Due to lack of space, not all equations have been displayed and long expressions had to be broken at the end of a line, any inconvenience due to this is regretted. The open square symbol denotes the end of a proof. The reference list is far from complete, we have often included only a recent or a representative paper. We apologize for any unintended exclusion of a reference.

It is a pleasure to remember here people who have contributed to the preparation of this book. Professor K. R. Parthasarathy was instrumental in introducing us to the subject and one of us (K. B. S.) has collaborated with him extensively over nearly two decades; without the insights and masterly expositions of him and of Professor P. A. Meyer, the subject may not have reached the stage it is in now. We thank all our friends, collaborators and members of the Q–P club who have helped us directly or indirectly in this endeavor. In particular, we must mention Professors Luigi Accardi, Robin Hudson, V. P. Belavkin, Martin Lindsay, Franco Fagnola, Stephane Attal, Jean-Luc Sauvageot, Burkhard Kümmerer, Hans Maassen, Rajarama Bhat and Dr Arup Pal and Dr Partha Sarathi Chakraborty. We are grateful to the Indian Statistical Institute (both Delhi and Kolkata campuses) for providing the necessary facilities, Indo-French Centre for the Promotion of Advanced Research and DST-DAAD agencies for making many collaborations possible. One of us (D. G.) would like to thank the Alexander von Humboldt Foundation for a postdoctoral fellowship during 2000–01 (and also later visits under its scheme of 'resumption of fellowship'), when part of the work covered by this book was done. We must also thank Dr Lingraj Sahu, who as a graduate student at a critical stage of writing the monograph, helped with introduction of a part of the material and Mr Joydip Jana for help with proofreading. One of the authors (D. G.) dedicates this book to his wife, Gopa and the youngest addition to his family, expected possibly before this book sees the light of the day; and acknowledges with gratitude the constant encouragement and support from his parents, mother-in-law and Amit-da during the writing of the book.

As is often the case in any such enterprise, some important topics (e.g. stop times) have been left out. The responsibility for the choice of topics as well as for any omissions and shortcomings of the text is entirely ours. We can only hope that this monograph will enthuse some researchers and students to solve some of the problems left unsolved.

K. B. Sinha
Debashish Goswami

Notation

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$I\!N$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}_+$	The set of nonnegative real numbers
$\mathbb{Z}$	The set of integers
S^1	The circle group
$\mathbb{T}^n$	The n-torus
C	The set of bounded continuous functions in $L^2(\mathbb{R}_+, k_0)$ or $L^2 \cap L^4_{\text{loc}}(\mathbb{R}_+, k_0)$
$\text{Dom}(T)$	Domain of an operator T
$\text{Ran}(T)$	Range of T
$\text{Ker}(T)$	Kernel of T
$\text{Sp}(A)$	Complex linear span of vectors in the set A
$\dim(V)$	Dimension of the vector space V
$\text{Im}(x)$, $\text{Re}(x)$	Imaginary and real parts of x (complex number or bounded operator)
$h, \mathcal{H}$	Hilbert spaces
$\text{Lin}(V_1, V_2)$	Set of linear maps from V_1 to V_2 (vector spaces).
$\mathcal{B}(\mathcal{H})$	The set of bounded linear operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}(\mathcal{H}, \mathcal{K})$	The set of bounded linear maps from $\mathcal{H}$ to $\mathcal{K}$ (Hilbert spaces)
$\mathcal{B}^{\text{s.a.}}(\mathcal{H})$	The real Banach space of all bounded self-adjoint operators on $\mathcal{H}$
$\mathcal{K}(\mathcal{H})$	The set of compact operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}_1(\mathcal{H})$	The complex Banach space of trace-class operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}_1^{\text{s.a.}}(\mathcal{H})$	The real Banach space of self-adjoint trace-class operators on $\mathcal{H}$
$\mathcal{L}(E, F)$	The set of adjointable maps from E to F (Hilbert modules)
$\mathcal{L}(E)$	The set of adjointable maps on a Hilbert module E

$\mathcal{K}(E)$	The set of compact adjointable maps on a Hilbert module E
$\mathcal{A}'$	The commutant of (C^* or von Neumann algebra) $\mathcal{A}$
$\mathcal{A}_+$	The set of positive elements of C^* or von Neumann algebra $\mathcal{A}$.
$\mathcal{M}(\mathcal{A})$	The multiplier algebra of $\mathcal{A}$
$\Omega, \Omega_{\mathcal{A}}$	The set of all states, of all normal linear functional on a C^*- or von Neumann algebra $\mathcal{A}$
$\otimes_{\text{alg}}$	Algebraic tensor product (between spaces or algebras)
$\Gamma(\mathcal{H})$	The symmetric Fock space over the Hilbert space $\mathcal{H}$
Γ	The symmetric Fock space over $L^2(\mathbb{R}_+, k_0)$ for some Hilbert space k_0
$\Gamma^f(\mathcal{H})$	The free Fock space over $\mathcal{H}$
k_t	The Hilbert space $L^2([0,t], k_0)$ for some Hilbert space k_0
k^t	$L^2((t,\infty), k_0)$
Γ_t	$\Gamma(k_t)$
Γ^t	$\Gamma(k^t)$
$e(f)$	The exponential vector $\oplus_{n=0}^{\infty} \frac{f^{\otimes^n}}{\sqrt{n!}}$
$\Gamma(A)$	The second quantization of A
χ_A	The characteristic function of the set A
f_t	The function $f\chi_{[0,t]}$
f^t	The function $f\chi_{(t,\infty)}$
Θ	The structure matrix
$\mathcal{U}_t$	Time reversal operator in Fock space
$\mathcal{A} >\!\!\lhd_\alpha G$	The crossed product of $\mathcal{A}$ (C^* or von Neumann algebra) by the action α of a group G
$a_R^\dagger(\cdot), a_\delta^\dagger(\cdot)$	The creation integrator processes associated with operator R and map δ respectively
$a_R(\cdot), a_\delta(\cdot)$	The annihilation integrator processes associated with operator R and map δ respectively
$\Lambda_T(\cdot), \Lambda_\sigma(\cdot)$	The number integrator processes associated with operator T and map σ respectively
$\mathcal{I}_{\mathcal{L}}(\cdot)$	The time integrator process associated with map $\mathcal{L}$ on $\mathcal{A}$
L^p_{loc}	The set of all $f \in L^2(\mathbb{R}_+, k_0)$ such that $\int_0^t \\|f(s)\\|^p ds < \infty$ for every $t \geq 0$
$L^p(\mathcal{A}, \tau)$	The noncommutative L^p spaces associated with trace τ

Contents

1
Introduction

The motivations for writing the present monograph are three-fold: firstly from a physical point of view and secondly from two related, but different mathematical angles.

At the present time our mathematical understanding of a conservative quantum mechanical system is reasonably complete, both from the direction of a consistent abstract theory as well as from the one of mathematical theories of applications in many explicit physical systems like atoms, molecules etc. (see for example the books [12] and [108]). However, a nonconservative (open/dissipative) quantum mechanical system does not enjoy a similar status. Over the last seven decades there have been many attempts to make a theory of open quantum systems beginning with Pauli [104]. Some of the typical references are: Van Hove [126], Ford *et al.* [52], along with the mathematical monograph of Davies [35]. The physicists' Master equation (or Langevin equation) was believed to describe the evolution of a nonconservative open quantum (or classical) mechanical system, a mathematical description of which can be found in Feller's book [50].

Physically, one can conceive of an open system as the 'smaller subsystem' of a total ensemble in which the system is in interaction with its 'larger' environment (sometimes called the bath or reservoir). The total ensemble with a very large number of degrees of freedom undergoes (conservative) evolution, obeying the laws of standard quantum mechanics. However, for various reasons, practical or otherwise, it is of interest only to observe the system and not the reservoir, and this 'reduced dynamics' in a certain sense obeys the Master equation (for a more precise description of these, see [35]). Since it is often impossible and impractical to solve the equation of evolution of the total ensemble, it is often meaningful to replace the reservoir by a 'suitable stochastic process' and couple the system with the stochastic process. In the case in which the stochastic process is classical, the total evolution can be described by a suitable stochastic differential equation (for an introduction to this, the reader is referred to [75] and [41]). The standard Langevin equation [52] involving the stochastic process should restore the conservativeness of the total system albeit for almost all paths. However, in many of the models studied by physicists this is not so.

The simplest quantum mechanical system is the so-called harmonic oscillator. However, the (sub-critically) damped harmonic oscillator which has been studied in classical physics since the time of Newton eludes a consistent treatment in conventional quantum mechanics. In the view of the present authors, this happens because the damped harmonic oscillator is a nonconservative, dissipative system and cannot be understood as a flow in a symplectic manifold (classical case) or in a standard Weyl canonical commutation relations (CCR) algebra (quantum case). One possible way to model this is to represent the environment or reservoir (responsible for the friction or the damping term) by an appropriate stochastic process, restore the unitary stochastic evolution of the quantum system and then project back to the 'system space' by 'washing out' the influence of the stochastic process (taking expectation with respect to the stochastic part) to get back the required nonconservative dynamics. This has been studied in [119] and has also been described in some detail in Chapter 7. Thus one can enunciate a philosophy, not too far away from that of the physicists, that given a nonconservative dynamics of a quantum system, one aim is to canonically construct the stochastic process which will represent the environment so that the two together undergo a conservative evolution and the projection to the system space restores exactly the nonconservative evolution. There is a further aim of the physicist, viz. to obtain the stochastic process mentioned above in a suitable approximation from the mechanical descriptions of the particles constituting the reservoir and of their interactions with the observed system. This aspect is not treated in this monograph and the reader is referred to [4], [8] and [35].

There is an exact mathematical counterpart to the picture in physicists' mind as described above. Given a finite probability space $S \equiv \{1, 2, \ldots, n\}$ with probability distribution given by the vector $p \equiv (p_1, p_2, \ldots, p_n)$ on it and a stochastic (or Markov) matrix $(t_{ij})_{ij=1}^{n}$ such that $t_{ij} \geq 0$, $\sum_{j=1}^{n} t_{ij} = 1$, one can associate a (discrete) evolution $(Tf)(i) = \sum_{j=1}^{n} t_{ij} f(j)$ with $f : \{1, 2, \ldots, n\} \to \mathbb{R}$. Then one observes that

(i) T maps positive functions f to positive functions and maps identity function to itself.
(ii) The probability distribution vector p is in one-to-one correspondence with the dual ϕ_p of the algebra of functions on S by $p \mapsto \phi_p$, where

$\phi_p(f) = \sum_{i=1}^{n} p_i f(i)$, and this induces a dual dynamics T^* given by $(T^*\phi_p)(\chi_j) = \sum_{i=1}^{n} p_i t_{ij}$, where χ_k denotes the characteristic function of the singleton set $\{k\}$.

(iii) $T^n, n = 0, 1, \ldots,$ and $T^{*n}, n = 0, 1, 2\ldots$ provide two discrete (dynamical) semigroups, the second being dual to the first; and clearly T^n for each n satisfies the property (i).

There is a standard construction of a Markov process (in this case Markov chain); see e.g. Feller's book [50]. This procedure extends naturally, beginning with the consideration of the algebra of functions on S as the algebra of $n \times n$ diagonal matrices and $\{T^n\}_{n=0,1,2,\ldots}$ as a positive semigroup on that, to the more general picture considering semigroups (discrete or continuous parameter) on the noncommutative algebra of all $n \times n$ matrices. What is perhaps surprising and is contrary to intuition in classical probability is that a very large class of Markov processes (including Markov chains) can be described by quantum stochastic differential equations in Fock space, again facilitating many computations ([99, 100]).

At this point an important generalization of the class of positive maps on an algebra makes its entrance. From a physical point of view, consider the following scenario. Let $\mathcal{H}$ be the Hilbert space of a localized quantum system A in a box and let there exist another quantum system B with associated Hilbert space $\mathbb{C}^n$. The system B is so far removed from A that there is no interaction between A and B and thus the Hilbert space for the joint system A and B is $\mathcal{H} \otimes \mathbb{C}^n$. Let T_n be the positive linear map which describes an operation on the joint system that does not affect B, given by $T_n(x \otimes y) = T(x) \otimes y$ for $x \in \mathcal{B}(\mathcal{H})$, $y \in \mathcal{B}(\mathbb{C}^n)$ (here $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on the Hilbert space $\mathcal{H}$ defined everywhere) for some positive linear map T on $\mathcal{B}(\mathcal{H})$. It seems reasonable to expect that given a positive linear map T on $\mathcal{B}(\mathcal{H})$, it should be such that for every natural number n, T_n given above should be positive. In such a case, T is said to be completely positive (CP) and such CP maps or semigroups of such maps play a very important role in the description of nonconservative dynamics on quantum systems. It is also useful to note that if the algebra involved is commutative (like the algebra of $n \times n$ diagonal matrices in the first example instead of $\mathcal{B}(\mathcal{H})$ or the whole matrix algebra) positivity and complete positivity are equivalent and that is why complete positivity does not surface in the context of nonconservative evolutions of classical physical systems. A detailed mathematical study of CP maps and of semigroups of CP maps on an algebra is done in Chapters 2 and 3, respectively.

As we had mentioned earlier in the context of a physical subsystem interacting with a reservoir in such a way that the reduced dynamics is governed by a

Master equation, it is natural to assume that the Master equation is just the differential form of a contractive semigroup of CP maps on the algebra describing the subsystem. Now we can turn this into a very interesting (and demanding) mathematical question: does there exist a 'suitable' probabilistic model for (a) the reservoir and for (b) its interaction with the given subsystem such that the expectation of the total evolution with respect to the probabilistic variables give the CP semigroup we started with? This is the general problem of 'dilation of a contractive semigroup of CP maps on a given algebra'. This problem is solved in Chapter 6 in complete generality under the hypotheses that the given semigroup of CP maps is uniformly continuous so that its generator acting on the given algebra is bounded.

There are complete descriptions of the structure of the generator of a uniformly continuous semigroup of CP maps on an algebra in the third chapter. Unfortunately the situation is far from settled for a similar question if the semigroup is only strongly continuous, which is, as is often the case, more interesting from the point of view of applications. However, if we pretend that the generator of the strongly continuous semigroup of CP maps on the algebra formally looks similar to that for the uniformly continuous case, then under certain hypotheses a class of strongly continuous semigroups can be constructed such that its generator coincides with the formal one on suitable domains. This is described in the second section of the same chapter along with an applications to a large class of classical Markov processes and also to the irrational rotation algebra which is a type II_1 factor von Neumann algebra. More details on these constructions and results on the unital nature of the semigroups, so constructed, can be found in Chebotarev [25]. This chapter ends with an important abstract theorem on noncommutative Dirichlet forms associated with a strongly continuous semigroup of CP maps on a von Neumann algebra equipped with a normal faithful semifinite trace. This result is then used in Chapter 8 to solve the dilation problem for such semigroups.

In order to carry out the program charted out in an earlier paragraph, it is necessary to develop some language and machinery. In Chapter 4, the basic theories of Hilbert C^*- and von Neumann modules and of group actions on them are presented. These ideas are then used to develop an elaborate theory of stochastic integration and quantum Itô formulae in symmetric Fock spaces extending the earlier theory as described in [97]. This language seems to be sufficiently powerful to allow a large class of unbounded operator-valued processes in Fock space to be treated. These methodologies were then used to solve Hudson–Parthasarathy (H–P)-type stochastic quantum differential equations with bounded coefficients (Chapter 5) and with unbounded coefficients (Chapter 7) giving unitary or isometric evolutions in a suitable Hilbert space as solutions. The Evans–Hudson (E–H)-type equation of observable or of an element of an algebra is re-interpreted as an equation on the space of maps on a suitable Fock Hilbert module and for bounded coefficient case, such equations are solved in Chapter 5. This language and associated machinery are important because they allow us to answer in the affirmative the problem of the dilation of a uniformly continuous semigroup of CP maps on an algebra.

Chapter 6 uses the tools of Chapters 4 and 5 to show that given a uniformly continuous semigroup of CP maps on a von Neumann algebra, there exists a quantum probabilistic model in the Fock space such that there is a E–H-type quantum stochastic differential equation describing the stochastic evolution of the observable algebra of the quantum subsystem coupled to the quantum stochastic process in Fock space modeling the reservoir, and such that the expectation gives back the original CP semigroup. This construction is canonical and interestingly gives a quantum stochastic differential equation for the evolution so that further computations for any other observable effects may be facilitated.

The mathematical problem of stochastic dilation of a semigroup of CP maps on a C^*- or von Neumann algebra, uniformly or strongly continuous, with the additional requirement that the dilated map on the algebra satisfies a quantum stochastic differential equation in Fock space and is a $*$-homomorphism on the algebra of observables is the central mathematical problem treated in this book. The property of $*$-homomorphism of such maps is a basic requirement of any quantum theory and the fact that these also satisfy a differential equation makes the family of dilated maps a stochastic flow of $*$-homomorphisms on the algebras. In fact, Chapters 6 and 8 are devoted to the final steps of the solution of this problem, the first for the uniformly continuous semigroup and the second for the strongly continuous one, while the Chapters 2 to 5 and Chapter 7 deal with preliminary materials and develop the machinery needed. This completes our discussions on the central mathematical problem treated here along with its connection to applications, arising from the physics of open quantum systems.

There is a another mathematical direction from which we approach the central mathematical problem of stochastic dilation, viz. that of noncommutative geometry. Chapter 9 should not be and cannot be thought of as an exposition on the rapidly developing subject of noncommutative geometry as created by Alain Connes [28] (the reader may also look at the books [82] and [56]). Instead, after some introduction to basic concepts in differential geometry and elements of noncommutative geometry, three explicit examples are worked out and in each case an appropriate associated stochastic process (classical or quantum) is constructed. Much more study in these areas remains to be done; for example one can investigate whether the nontrivial curvature in the Quantum Heisenberg manifold can be captured in terms of the stochastic processes on it.

We think the spirit of the book is perhaps well-described in the preface by Luigi Accardi in *Probability Towards 2000* [3] and we quote:

> The reason why the interaction of probability with quantum physics is different from the above mentioned ones is that the problem here is not only to apply classical techniques or to extend them to situations which, being even more general, still remain within the same qualitative type of intuition, language and techniques. Furthermore, the formalism of quantum theory, with its complex wave functions and Hilbert spaces, operators instead of random variables, creates a distance between the mathematical model and the physical phenomena which is certainly greater than that of classical physics. For these reasons, these new languages and techniques might be perceived as extraneous by some classical probabilists and researchers in mathematical statistics. However, the developments motivated by quantum theory provide not only powerful theoretical tools to probability, but also some conceptual challenges which can enter into the common education of all mathematicians in the same way as happened for the basic qualitative ideas of non-Euclidean geometries.