ELEMENTS OF COMPLEX ANALYSIS

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ADDISON-WESLEY PUBLISHING COMPANY
Reading, Massachusetts • Menlo Park, California • London • Don Mills, Ontario

This book is in the
ADDISON-WESLEY SERIES IN MATHEMATICS

Consulting Editor:
Lynn H. Loomis

Library of Congress Catalog Card No. 71-76071.

Preface

The relatively advanced age of complex function theory and the resulting codification of the methods employed therein, together with the large number of distinguished mathematicians who, over many decades, have devoted themselves to expository works on the subject, have largely precluded essential variations in the material covered or the techniques employed in introductory complex analysis texts. As a result the major considerations of an author in this field revolve about style, lucidity, the choice of introductory material to equalize the disparity of backgrounds of potential readers, and the desire to include in restricted space a reasonable amount of complex analysis per se without glossing over the difficulties (mostly associated with the topology of the plane) which are inherent in the foundations of the theory. The last desideratum presents the greatest challenge. Of the several ways of coping with it (e.g., introducing combinatorial techniques, basing everything on winding number, ignoring it), we have chosen one which we hope to be acceptable to the reader (or teacher) who is not disposed to worrying with such considerations in the study of complex analysis, as well as to the purist. Namely, of the difficult plane topology, we have accepted the Jordan curve theorem without proof and have based essentially all other considerations upon it and results whose proofs are provided. On this basis we give (Section 17) a non-intuitive, geometrically motivated (and admittedly lengthy), discussion of the orientation of a simple closed curve relative to its embedding in the plane, which leads in a simple way to a non pictorial proof of Cauchy's formula for simple closed paths. On the other hand, this discussion is omittable (or its intuitive content quickly perceived) without loss of continuity; and so are the remaining sections marked *, which depend upon Section 17.

Five sections (marked †) are concerned with the winding number and related matters. This concept, which we regard as enlightening but not basic to our treatment, is not used in other sections. Section 15, which contains a geometric discussion of the argument which is more careful than that sometimes found in elementary texts, also is not prerequisite to any other sections.

The teaching of complex analysis at the undergraduate level is very similar in content, if not in rapidity of development, to that of a beginning graduate course. Accordingly, we have included sufficient background material concerning the real number system and the rudiments of metric spaces to make the book self-contained and hence suitable at the senior or graduate level. Moreover, we have tried to supply explanations and proofs in detail keeping with the level of sophistication of the undergraduate reader. There are many gnawing little things which are sometimes left unsaid in textbooks, things which a reader of some mathematical experience takes for granted but which are worrisome to the conscientious student and overlooked entirely by his less demanding (but more frequent) peers. Neither do we seem to offend graduate students by inclusion of such items.

Most of the material of Chapter 1 will be in the repertoire of students who have had courses in advanced calculus, but by setting out the needed facts as theorems and proving them we hope to remove from the instructor of a class whose members have varying backgrounds the burden of reviewing all of these facts in detail during class time. In our opinion a selective sample of the important proofs of sections 1–10 is optimal for most courses.

Sections 1 to 58 should provide more than enough material for a one semester graduate course or a two quarter undergraduate course. In such a program one would probably omit all sections marked *, †, or ‡ as well as 6.4–6.9, 8.8–8.11, 12.10, 12.11, 12.13–12.16, 12.19–12.21. The last four chapters are essentially independent of each other.

In addition to their usual purpose or stimulating the reader to interact with the mathematics, the exercises serve as places to outline (through the hints) propositions which are needed in later developments but whose proofs are cumbersome, or similar to others presented, or otherwise unsuited for detailed discussion in the text. All such exercises, as well as certain others, are solved in *Solutions to Selected Exercises.*

The bibliography contains specific references from the text, while the collateral reading list is limited to items we have found most helpful or interesting. No attempt is made to justify the myriad plagiarisms.

Finally, a word to the student. There are many allusions to "rigor" and "intuition" throughout the book, and perhaps you will feel their purpose is for lauding the former and deprecating the latter. On the contrary, each has an important place in mathematics. Moreover, they are not absolutes; proofs which are considered rigorous, that is, acceptable, by one generation often lose that status with the passage of time and, occasionally, vice versa. In fact, proof techniques which are accepted by certain mathematicians may be rejected by their contemporaries. Thus the decision whether or not a proof is rigorous seems to be largely determined by intuition tempered by training; in any event it is clearly a subjective decision. We feel, however, that it is important that an introductory text at this level make clear which portions of a discussion are simply explanatory and which arguments are likely to be regarded as rigorous by the practicing analyst. Maintaining rigor in this sense means, roughly, that proofs based on visual reasoning, via identification of the complex numbers with points of a plane, are to be avoided. In keeping with this viewpoint, this book is not concerned with the subtleties of mathematical logic, axiomatic set theory, etc.

You may find certain sections (notably 12, 15, 17, 18) more dull than difficult. Rapid initial reading of these for general content and returning to them when you have progressed sufficiently far to appreciate their purposes may help to alleviate your impatience to get to the core of the subject.

The tendency to become bogged down in passively checking the details of a proof can be countervailed by actively seeking the guiding idea (i.e., the intuitive part) and then checking that all niceties have been correctly attended to. A more vigorous (and highly recommended) approach is to read the assertion and then attempt to supply your own proof. Even if you fail you will gain understanding of the result and appreciation for the proof.

J.D.D.
C.C.O.

January 1969

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CHAPTER 1

PRELIMINARY CONCEPTS

1. SETS AND FUNCTIONS

1.1 Two concepts that are fundamental to mathematics, and particularly to analysis, are those of set and function. Indeed, our primary interest is the study of functions defined on a particular set, the set of complex numbers. A description adequate for our purposes is that a *set* is a collection of identifiable objects. Most of the sets that we will consider will be sets of numbers of some kind, but the importance of sets in mathematics is associated with the generality, and hence the flexibility, of the concept. Thus there is no restriction on the kinds of "objects" that may constitute a set. The terms "family" and "collection" are sometimes used instead of "set", particularly when the objects of the set are themselves sets.

We use the notation $a \in A$ to indicate that a is one of the objects in the set A, and we say that a is an element of A or that a belongs to A. If a is not an element of A, we indicate this fact by $a \notin A$. Often sets are defined as consisting of all objects that possess a certain "property", say P. This is indicated concisely by the notation $A = \{a : a \text{ satisfies } P\}$. (The colon might be read "such that".) In this notation the set of odd integers, for example, could be designated as $A = \{a : a = 2k + 1, k \text{ is an integer}\}$. The set consisting of the objects $a_1, \ldots, a_n$ is denoted by $\{a_1, \ldots, a_n\}$; in particular, the set consisting of the sole element a is denoted by $\{a\}$.

A set A is called a *subset* of set B, written $A \subset B$, if every element of A is also an element of B. (We also write $B \supset A$ to mean the same thing.) Two sets are called *equal*, written $A = B$, if $A \subset B$ and $B \subset A$, that is, if the sets contain precisely the same elements. Thus a set is determined solely by the elements that it contains, or to phrase it another way, a set A is considered to be defined if, for every object, x, it is possible to determine whether or not $x \in A$. (Assigning a suitable meaning to "determine" is admittedly a thorny (and unsettled) mathematical problem.) The fundamental way to prove that $A = B$ is to prove that every element of A is an element of B, and vice versa. However, there are a number of formal rules for manipulating sets which generally are helpful in establishing that one set is a subset of another or that two sets are equal. Some of these rules appear in the exercises.

It is expedient to introduce a "set" that contains no elements. It is called the *empty set* and is denoted by $\varnothing$. Thus for every set A, $\varnothing \subset A$.

If A and B are two sets, the set of all elements that belong either to A or to B is called the *union* of A and B and is denoted by $A \cup B$. Thus

$$A \cup B = \{a : a \in A \text{ or } a \in B\}.$$

The set of elements that belong to both A and B is called the *intersection* of A and B and is denoted by $A \cap B$. Thus $A \cap B = \{a : a \in A \text{ and } a \in B\}$. Sets A and B are called *disjoint* if $A \cap B = \varnothing$. By the *difference* $A - B$ we mean those elements of A that are not elements of B; that is, $A - B = \{a : a \in A \text{ and } a \notin B\}$. Frequently in the course of a discussion it is understood that all sets under consideration are subsets of some "universal" set X. In this case the difference $X - A$ is referred to as the *complement* of A (with respect to X).

1.2 An *ordered pair* (a, b) is an object associated with the individual objects a and b, in which the order in which a and b are written is considered essential to the identity of (a, b). Therefore, unless $a = b$, (a, b) and (b, a) are distinct ordered pairs. (This is, of course, in contrast to the sets $\{a, b\}$ and $\{b, a\}$, which are equal.) Thus two ordered pairs, (a, b) and (c, d) are equal if and only if* $a = c$ and $b = d$.

If A and B are nonempty sets, the *cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) for which $a \in A$ and $b \in B$; that is $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. The reader should observe that this is a generalization of the way in which points of a plane are represented as pairs of real numbers; that is to say, a plane can be interpreted as the cartesian product of two lines.

1.3 The reader is familiar with the concept of a function as a rule which associates with each element of a set, say A, a *unique* element of a set B. This description of function leaves much to be desired, however, since essentially it merely substitutes for the term to be defined another word, namely, "rule". While the reader must never lose sight of the intuitive idea embodied in this description, he should also realize that the term "function" can be defined in a

*The expression "if and only if", which we will henceforth abbreviate iff, occurs frequently in mathematical exposition. If P and Q are two propositions, then "if P (is true), then Q (is true)" means that P implies Q, whereas "Q (is true) only if P (is true)" means that Q implies P. Thus the assertion "P (is true) iff Q (is true)" means that P and Q are logically equivalent propositions. A similar terminology centers about the terms "necessary" and "sufficient". If P implies Q we say that P is sufficient for Q, whereas if Q implies P we say that P is necessary for Q. Thus "necessary and sufficient" means the same thing as "if and only if", with the necessary modifications occasioned by grammatical demands.

precise and concise way by use of the idea of cartesian product. The number of undefined terms is thereby kept to a minimum. The definition is motivated by the relationship between a function and its graph, our viewpoint being that a function is abstractly identical with its graph, which is defined to be a special kind of subset of $A \times B$. Specifically, a nonempty subset f of $A \times B$ is called a *function* (from A into B) if $(a, b) \in f$ and $(a, b') \in f$ together imply $b = b'$. Thus when f is specified, b is uniquely determined by a; this fact is customarily denoted by $b = f(a)$. Here b is called the *value* of f at a. Note that our definition does not require that every element of A appear as first member of an ordered pair belonging to f. Those elements of A which do so appear constitute the *domain* of f, designated D_f. To repeat,

$$D_f = \{a : (a, b) \in f \text{ for some } b \in B\}.$$

(Some authors, in defining "function from A into B", make the definition such that $D_f = A$.) Similarly, the *range* of f is the subset of B defined by $R_f = \{b : (a, b) \in f \text{ for some } a \in A\}$. If it happens that $R_f = B$, this fact can be indicated by the rather subtle change of terminology, "f is a function from A *onto* B".

Synonyms for "function" are *transformation*, *mapping*, *map*, *operator*, and *correspondence*. Observe that f is the symbol for the function, whereas $f(a)$ is an element of R_f. It is generally prudent to distinguish these notations, although in complex analysis the distinction is honored more in the breach than in the observance. Thus the symbol $f(a)$ is often used simultaneously to name the function and to indicate that a is the symbol for a "general" element of D_f. This usage has certain advantages and is widely understood; so, in the interests of simplicity and clarity, in later sections we will stray without apology from the logically correct notation. If f and g are functions from A to B and $f \subset g$, then f is said to be a *restriction* of g (to D_f), or g is called an *extension* of f. This means that $D_f \subset D_g$ and $f(a) = g(a)$ for all $a \in D_f$.

A simple yet very important function is the identity function I. For this function D_I can be any specified subset of A, and $R_I = D_I$. The function I is then defined by $I = \{(a, a) : a \in D_I\}$, or alternatively by the specification $I(a) = a$ for every $a \in D_I$. There are, of course, different identity functions corresponding to different choices of domain, but it is in general satisfactory to use the symbol I for each of them.

1.4 Let f be a function from A into B. If $C \subset A$, then $f(C)$ is the subset of B defined as $f(C) = \{b : b = f(c) \text{ for some } c \in C\}$. Thus, for example, $f(A) = f(D_f) = R_f$. Similarly, if $D \subset B$ we define $f^{-1}(D)$ to be the subset $f^{-1}(D) = \{a : f(a) \in D\}$. Therefore $f^{-1}(B) = f^{-1}(R_f) = D_f$; and when D is a set consisting of a single point b, $f^{-1}(b)$ consists of those $a \in D_f$ for which $f(a) = b$. Observe that $f^{-1}(b) = \varnothing$ iff $b \in B - R_f$.

1.5 If f is a function having the property that for each $b \in R_f$, $f^{-1}(b)$ contains only one element, then f is said to be one to one. It is clear that in this case (and only in this case) f^{-1} defines a function (call it f^*) with domain R_f and range D_f. More precisely, f^* is the subset of $B \times A$ defined by $(b, a) \in f^*$ iff $(a, b) \in f$. The one-to-one property makes this a function. To state this yet another way, $a = f^*(b)$ iff $b = f(a)$. We shall follow common usage and denote this *inverse* function f^* by the symbol f^{-1}, though it should not escape the reader's attention that we use this notation in two different senses.

As an illustration consider the function f whose domain D_f is the set of all real numbers and whose value at each real number a is the real number a^2. (The real numbers are discussed in the next section.) Its range R_f will then be the set of nonnegative real numbers. The function f is not one to one since, for example, $f(-2) = f(2) = 4$. Moreover, $f^{-1}(4) = \{-2, 2\}$, a set of two elements. The function f does not have an inverse function. Consider, by comparison, the function g whose domain D_g is the set of all nonnegative real numbers and whose value at each real number a is a^2. Its range R_g is the set of nonnegative real numbers. The function g is one to one and has an inverse function; for example, $g^{-1}(4) = 2$.

The geometric interpretation of the distinction between f and g should not go unnoticed. The fact that f does not have an inverse is associated with the fact that certain horizontal lines have more than one point in common with the (parabolic) graph (of) f, but this is not the case for the (half-parabolic) graph (of) g. (See Fig. 1.)

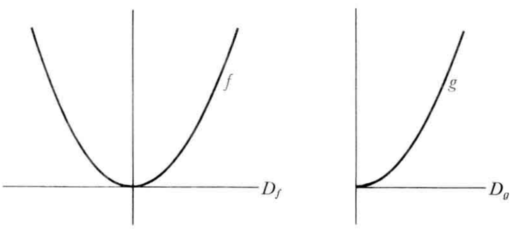

Fig. 1

A one-to-one function f is sometimes said to establish a *one-to-one correspondence* between D_f and R_f.

In discussing functions it is customary to use expressions such as "f is a real (valued) function defined on E" to mean that R_f is a subset of the set of real numbers and that $D_f = E$.

1.6 Suppose that f is a function from A into B and g is a function from B into C such that $D_g \supset R_f$. Then the *composition* of g and f is a function, designated by $g \circ f$ (in that order), from A into C, with domain D_f and defined by $g \circ f = \{(a, c) : (a, b) \in f \text{ and } (b, c) \in g \text{ for some } b \in B\}$. In other words, the value of

$g \circ f$ at a is $(g \circ f)(a) = g(b) = g\big(f(a)\big)$. It is easily seen that if f is one to one, then $f \circ f^{-1} = I$ and $f^{-1} \circ f = I$ (where in the former equality I is the identity function with domain $D_I = R_f$, and in the latter I is the identity function with $D_I = D_f$).

When the domain of a function f is itself a subset of a cartesian product $E \times K$, the value of f at the point $(e, k) \in E \times K$ is denoted by $f(e, k)$ rather than $f\big((e, k)\big)$.

1.7 We introduce one further concept. If M is a nonempty set and to each $\mu \in M$ there corresponds a set A_μ, then the set M is called an *index set* for the *family* of sets $\{A_\mu : \mu \in M\}$. We define unions and intersections of the sets of the family as

$$\bigcup_{\mu \in M} A_\mu = \{a : a \in A_\mu \text{ for some } \mu \in M\}$$

and

$$\bigcap_{\mu \in M} A_\mu = \{a : a \in A_\mu \text{ for every } \mu \in M\}.$$

Even when no specific index is mentioned for a family of sets, we still use the symbols $\cup$ and $\cap$ to indicate the union and intersection of the sets of the family, with minor variations in notation, the meanings of which will be obvious in each case.

EXERCISES 1

1. Describe in words the meaning of each of the following relations between two sets:
   $A \cap B = \varnothing$, $A \cup B = \varnothing$, $A \cap B = A$, $A \cup B = A$, $A \cup B \subset A$, $A \cap B \subset A$.
2. Prove that $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (If $\cup$ and $\cap$ are associated with "plus" and "times", the first three formulas are obvious analogues of formulas you have seen before.)
3. Show that $A \cap B$ and $A - B$ are disjoint and that $A = (A \cap B) \cup (A - B)$.
4. Prove the *DeMorgan* rules: $X - (A \cup B) = (X - A) \cap (X - B)$ and $X - (A \cap B) = (X - A) \cup (X - B)$. These are valid for arbitrary sets X, A, B, but when X is the universal set of 1.1, these rules can be stated in a particularly simple form; e.g., the first asserts that the complement of a union is the intersection of the complements.
5. Extend DeMorgan's rules to families of sets.
6. If $A = A_1 \cup A_2$, then $A \times B = (A_1 \times B) \cup (A_2 \times B)$. If $C = A_1 \cap A_2$, then $C \times B = (A_1 \times B) \cap (A_2 \times B)$.

7. Define the *symmetric difference* of A and B, designated $A \Delta B$, by $A \Delta B = (A \cup B) - (A \cap B)$. Show that $A \Delta B = B \Delta A$ and that $A \Delta B = (A - B) \cup (B - A)$. Describe the points in $A \Delta B$ according to each of these formulas, and convince yourself on this basis that the two formulas represent the same set. Show that $A \Delta A = \varnothing$ and $A \Delta \varnothing = A$; explain on the basis of the description you gave of $A \Delta B$. Show that $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$. Explain.
8. Given that f is a function, find the relationship between
   (a) $f(A \cup B)$ and $f(A) \cup f(B)$,
   (b) $f(A \cap B)$ and $f(A) \cap f(B)$,
   (c) $f^{-1}(A \cup B)$ and $f^{-1}(A) \cup f^{-1}(B)$,
   (d) $f^{-1}(A \cap B)$ and $f^{-1}(A) \cap f^{-1}(B)$,
   (e) $f(X - A)$ and $f(X) - f(A)$, where $A \subset X = D_f$,
   (f) $f^{-1}(X - A)$ and $f^{-1}(X) - f^{-1}(A)$, where $A \subset X = R_f$.
9. Let f be a function. In what sense is it true that $f \circ f^{-1} = I$ and that $f^{-1} \circ f = I$?
10. Let the functions f, g satisfy $g \circ f = I$, where $D_I = D_f$. Show that f has an inverse function. Does it follow that $g = f^{-1}$?
11. In addition to the conditions given in Exercise 10, suppose $f \circ g = I$, where $D_I = D_g$. Show that $g = f^{-1}$.
12. Prove that $f \circ (g \circ h) = (f \circ g) \circ h$ whenever both sides are defined.
13. Phrase the definition of index set in terms of function.

2. EQUIVALENCE RELATIONS

2.1 A natural generalization of the concept of function is that of *relation*, which is defined to be an arbitrary subset of $A \times B$. The idea here is that we wish to "relate" certain elements of A to certain elements of B, where, unlike the situation for a function, an element $a \in A$ may have several correspondents in B. When we say that a subset r of $A \times B$ is a relation from A to B we are making specific the intuitive concept that those elements $b \in B$ that are related to a given $a \in A$ by our particular relation r are those for which $(a, b) \in r$. The relation concept is a very general one and embraces many familiar mathematical concepts (for instance, function, inequality of real numbers, similarity of triangles). However, there is just one type of relation other than functions which will concern us, namely, the equivalence relation.

2.2 A relation r from A to A is called an *equivalence relation* on A if it satisfies

i) $(a, b) \in r$ implies $(b, a) \in r$ (symmetry);

ii) $(a, a) \in r$ for every $a \in A$ (reflexivity);

iii) If $(a, b) \in r$ and $(b, c) \in r$, then $(a, c) \in r$ (transitivity).

2.3 In working with an equivalence relation it is customary to use a symbol somewhat resembling an equality sign to designate the relation between elements. Thus we shall write $a \approx b$ (read "a is equivalent to b") to mean $(a, b) \in r$. (Of course, when different equivalence relations appear in a discussion, we must use different symbols.) Thus the essential properties of $\approx$ are

i) $a \approx b$ implies $b \approx a$;

ii) $a \approx a$ for all $a \in A$;

iii) $a \approx b$ and $b \approx c$ imply $a \approx c$.

It is clear that the congruence of triangles or the similarity of triangles from geometry are examples of equivalence relations on the set of all triangles. Ordinary equality (identity) is perhaps the most familiar equivalence relation, and in a sense every equivalence relation can be interpreted as a kind of abstract equality. By way of illustration consider the concept of congruence of two triangles, a and b, where we say that a is congruent to b if a can be moved into coincidence with b by a rigid motion that does not alter the shape or size of a. (We need not worry here about giving a precise meaning to this last phrase.) Clearly, the intuitive idea that is meant to be conveyed by congruence of triangles is that they are "abstractly identical", differing only in their location. In this sense then, congruence of triangles might be considered equality, with the agreement that we choose to regard as irrelevant the location of a triangle in space. In fact, some geometry books refer to congruent triangles as "equal", though clearly they are not indistinguishable if they are not at the same place. Likewise, similarity of triangles can be thought of as equality, as determined by someone who chooses to neglect both the location and the size of a triangle and to consider only its shape.

2.4 Situations similar to these are often encountered in mathematics. A set A of distinguishable elements is given, but for certain considerations it is desirable to overlook some of the distinguishing features of the elements and to consider certain elements of A as being "abstractly identical". Let us call them equivalent instead of abstractly identical, and if a is equivalent to b, write $a \approx b$. Then our intuitive idea of what "abstractly identical" should mean demands that $\approx$ satisfy conditions (i), (ii), (iii).

On the other hand, suppose a definite equivalence relation $\approx$ is defined on a set A. Call a subset B of A an *equivalence class* for the relation $\approx$ if B consists of all elements of A that are equivalent to some specific element of A. Because of (ii), $B \neq \varnothing$. It is also apparent by (i) and (iii) that every element of B is equivalent to every other element of B and that no element of B is equivalent to an element of $A - B$. Thus the collection of all equivalence classes is a collection of nonempty pairwise disjoint sets whose union is A.

Moreover, two elements of A are equivalent if and only if they belong to the same equivalence class. We might say roughly that we have lumped together (into one equivalence class) all equivalent elements. Often we find it convenient to regard the equivalence classes as entities and to abstract ideas about them. In this way an equivalence relation on a set A leads to "abstract identification" of equivalent elements.

EXERCISES 2

1. Let q be a fixed integer. Call two integers m and n congruent if $m - n$ is divisible by q. Show that congruence is an equivalence relation on the set of integers and describe the equivalence classes.
2. Let f be a function from A into B, with $D_f = A$. If $a_1, a_2 \in A$, declare a_1 to be equivalent to a_2 iff $f(a_1) = f(a_2)$. Show that this defines an equivalence relation on A. Let A^* be the set of equivalence classes. Define a function f^* on A^* by $f^*(a^*) = f(a)$ iff $a \in a^*$. Prove that this stipulation actually defines a function, that $D_{f^*} = A^*$, that $R_{f^*} = R_f$, and that f^* is one to one.
3. The symbol $\leq$ defines a relation on the set of integers. Describe a set of properties of $\leq$ which you feel might be appropriate to generalize this relation to a general set. (Compare 2.2.) Give some examples of relations that have the properties that you have specified. Do the same for $<$.
4. Why is it incorrect to reason that, in 2.2, (ii) is a consequence of (i) and (iii) because (i) implies that the hypothesis of (iii) is valid when $c = a$?

3. REAL NUMBERS

Our primary purpose in this chapter is to define and study the set of complex numbers. The set of real numbers is of fundamental importance to this study, so we will briefly review its properties. Our procedure will be first to state a set of postulates which characterize the set of real numbers. With this accomplished we will develop some of the properties of the real numbers which will be useful in our consideration of the complex numbers. We first describe accurately a kind of mathematical system in which the familiar rules are valid for performing arithmetical operations.

3.1 Definition. *A field F is a set that satisfies the following postulates.*

The postulates of addition:

- $\boldsymbol{A}_1$. *To every two elements $a, b \in F$ there corresponds a unique element of F, called the sum of a and b and designated $a + b$.*
- $\boldsymbol{A}_2$. *Addition is commutative: $a + b = b + a$.*
- $\boldsymbol{A}_3$. *Addition is associative: $a + (b + c) = (a + b) + c$.*
- $\boldsymbol{A}_4$. *There exists a unique element $0 \in F$ such that $a + 0 = a$ for every $a \in F$.*