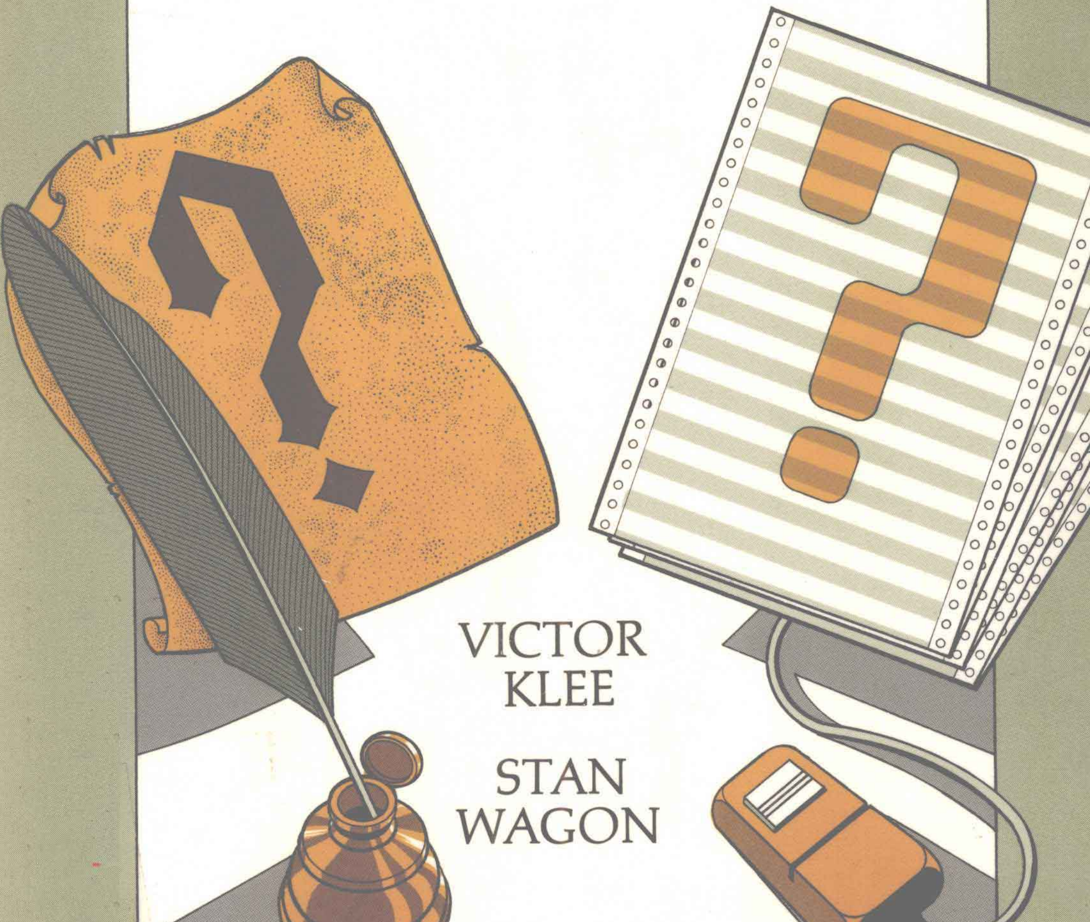
OLD AND NEW UNSOLVED PROBLEMS

in Plane Geometry and Number Theory



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STAN WAGON

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The Association, for its part, was delighted to accept the gracious gesture iniating the revolving fund for this series from one who has served the Association with distinction, both as a member of the Committee on Publications and as a member of the Board of Governors. It was with genuine pleasure that the Board chose to name the series in her honor.

The books in the series are selected for their lucid expository style and stimulating mathematical content. Typically, they contain an ample supply of exercises, many with accompanying solutions. They are intended to be sufficiently elementary for the undergraduate and even the mathematically inclined high-school student to understand and enjoy, but also to be interesting and sometimes challenging to the more advanced mathematician.

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PREFACE

THE ROLE OF UNSOLVED PROBLEMS

As mathematics becomes ever more complex, more structured, and more specialized, it is easy to lose sight of the fact that the research frontiers still contain many unsolved problems that are of immediate intuitive appeal and can be understood very easily, at least in the sense of understanding what a problem asks. Parts of plane geometry and number theory are especially good sources of such intuitively appealing problems, and those are the areas emphasized in this book. For each of the presented problems, we look at its background, try to explain why some mathematicians have found the problem of special interest, and tell about techniques that have been used to obtain partial results. Some of the exercises will help out readers establish partial results for themselves, and references are provided for those who want to learn more about the problems.

All mathematical discoveries may be said to consist of the solution of unsolved problems. Sometimes a question is answered almost as soon as it is formulated. Then the question and its answer are published together, and the question never becomes known as an unsolved problem. In other cases, a mathematical question may be formulated long before anyone is able to answer it. If the question becomes widely known, many researchers may direct their efforts toward answering it and the unsolved problem may have a significant influence on the development of mathematics.

Some unsolved problems are conceptually very simple and specific—for example, "Is the answer 'yes' or 'no'?", "What is the value of this function at a specified point of its domain?" Others are conceptually sophisticated—for example, "How can such and such a theory or argument be extended so as to apply to a certain more general class of objects?" Both the simple and the sophisticated problems have played an important role in the development of mathematics, and often they are inextricably mingled. Some of the "simplest" unsolved problems, such as Fermat's last "theorem" (see Section 13) have led directly to some of the most sophisticated mathematical developments and, in turn, to sophisticated unsolved problems. On the other hand, the solution of a sophisticated problem often requires answering a number of simple, specific questions. Thus it is probably unreasonable to try to decide whether the simple or the sophisticated problems have been more important in the development of mathematics. In any case, this book is devoted exclusively to problems of the simple sort—ones whose statements are short and easy to understand.

Some of the more modern areas of mathematics are poor sources for the sort of unsolved problem that is emphasized here. If one thinks of mathematical disciplines as buildings, some could be represented by buildings that are very tall but also quite narrow, with a few important research problems sprouting out of the roof. In order to reach these unsolved problems, one must climb through the many lower floors of the building, and on each floor must make one's way through numerous definitions and theorems that may be necessary for even understanding what the unsolved problems are about. By contrast, we are concerned here with some parts of mathematics that can be represented as broad, low buildings, with a high ratio of roof area (research problems) to volume. Any particular problem can be reached, at least in the sense of understanding what it asks, by entering the building through the proper door and climbing only one or two flights of stairs. It is striking that plane geometry and elementary number theory, the oldest branches of mathematics, still offer such a large supply of easily understandable, intuitively appealing, unsolved problems. This book contains some of our favorite examples from each subject.

Before proceeding to the problems themselves, we should admit that our picture of branches of mathematics as buildings is misleading in some important respects. For example, we have pictured elementary number theory as a broad, low building, and many mathematicians would think of the subject of algebraic geometry as one of the tallest, most highly structured buildings. However, algebraic geometry has provided some of the most important tools for dealing with some of the most "elementary" problems of number theory, such as Fermat's last theorem. This illustrates the intermingling of simple problems and sophisticated problems, and the important distinction between understanding a problem in the sense of having an intuitive grasp of what it asks, and the much deeper understanding that requires knowing enough to have some chance of solving the problem or at least discovering new results that are relevant to it. For each problem in this book, we hope to have provided enough information to enable the reader to understand the problem in at least the first sense. It remains to be seen what is required for understanding in the second sense. However, we don't believe that all of the problems presented here will require sophisticated methods for their solution. Probably some will be solved by means of a single really clever idea. There's no monopoly on clever ideas, and your chance of solving some of the problems may be as good as anyone's!

Part of the appeal of elementary problems is the feeling that they ought to have solutions that are equally elementary. But elementary problems can have exceedingly complex solutions. A notorious example is the four color theorem (every map in the plane can be colored with four colors so that adjacent countries are colored differently); its statement is simple but its proof, discovered in 1976 by K. Appel and W. Haken, is long, subtle, and involves an extensive computer calculation. Another example is a less well-known result due to P. Monsky: It is impossible to dissect a square into an odd number of triangles having equal area (see Section 7 for a discussion of this result and its extensions). The only known proofs use a sophisticated concept from field theory. Mathematics is full of simple-sounding statements that have been proved, but whose proofs are complicated, using the most sophisticated tools mathematicians know. But that's not the worst of it. It may be that for certain elementary conjectures, *no proof of either the conjecture or its negation exists.* Most interesting questions deal with infinitely many instances; indeed, much of the power and beauty of mathematics consists of finding a finite proof that answers infinitely many instances of a question. As observed by Haken[1] in an interesting paper that contains both details of the four color theorem's proof and some philosophical observations, it may be that "infinitely many single statements of a conjecture have infinitely many individual and significant differences so that a proof of the conjecture is impossible since it would require infinitely many case distinctions." (For more on the possibility that simple problems are undecidable, see Section 20.) Thus amateur and professional mathematicians are in comparable situations. An amateur might spend years searching for an elementary solution to an elementary unsolved problem, confident that such a solution exists. A professional, armed with supercomputers and the cumulative technical knowledge of generations of brilliant mathematicians, might attack a problem confident that a solution, perhaps a complicated one, exists. But both run the danger of being stymied by the nonexistence of the objects of their search.

Although we believe that most of our problems do have solutions, we would not be surprised if, for most of them, the solutions seem much less elementary than the problem statements. However, some problems that we expect to prove intractable may turn out to be solvable after all. For example, among the geometric problems that we originally planned for inclusion, both of us might well have cho-

[1] "An attempt to understand the four color problem," *Journal of Graph Theory,* 1 (1977) 193–206.

sen the main problem of Section 9—the modern form of the problem of squaring the circle—as the one least likely to be solved in the near future. But when the book was nearly completed, that problem was settled by M. Laczkovich, who proved that a circle (with interior) is equidecomposable to a square. His proof is quite complicated, using tools that on first glance would seem to have little to do with the problem.

ORGANIZATION OF THE BOOK

Each section of the book is devoted to a single problem or a related group of problems, and each section is divided into two parts. The first part is elementary. Its main goal is to supply the necessary definitions, convey some intuitive feeling for the problem, and present some easy proofs of relevant results. The second part contains more advanced arguments and mentions references. We hope that all of each first part and some of each second part will be accessible to undergraduate mathematics majors, and that everything in the book will be accessible to all graduate students of mathematics. On the other hand, since the presented problems are unsolved, and since fairly extensive references are supplied, there may be some interest in the research community as well.

The sections are arranged in three chapters. In each chapter, the first parts are grouped together, followed by the second parts. Thus the reader may peruse an entire collection of first parts without being distracted by technical details or references, and may then consult the second parts and the references for more information about whichever problems seem most appealing.

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and Joseph Zaks. Of course, any errors are our own, and we hope that readers will write to tell us about them.

Finally, we want to record our special thanks to the MAA's Beverly Ruedi, who took our jumbles of computer files and turned them into a book. It was a pleasure to work with her.

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and

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CONTENTS

CHAPTER 1

TWO-DIMENSIONAL GEOMETRY

INTRODUCTION

Of the ideas that lead beyond the geometry of Euclid, the notion of a convex set may be the one that would have been most easily understood by the ancient Greeks. In fact, a close relative of this notion was defined by Archimedes. A set is *convex* if it includes, with each pair of its points, the entire line-segment joining them. (Intuitively, a convex set is one that has neither holes nor dents.) As a source of appealing, easily stated unsolved problems, the study of low-dimensional convex sets rivals number theory and graph theory. Thus it is not surprising that convex sets play a central role in many of the sections of this chapter. However, there are also problems that don't involve convexity, and some of them involve modern (though elementary) set-theoretical and topological notions.

In the title of this book, the term "plane geometry" was used for brevity. However, that may suggest an axiomatic approach, and our approach here is certainly not axiomatic. Rather, our viewpoint is that the geometric objects really exist, and we want to study them in an efficient way. For that purpose, the most appropriate setting seems to be the cartesian plane $\mathbb{R}^2$, complete with its usual topology and basic vector operations. The title of this chapter, Two-Dimensional Geometry, is intended to suggest that, although all of the main discussion is in $\mathbb{R}^2$ and the various Parts One are confined to $\mathbb{R}^2$, many of the problems have interesting higher-dimensional analogues that are discussed briefly in Part Two.

In each section, the "title problem" is the one that begins the section and is stated in the Table of Contents. However, most of the sections include a variety of other, equally interesting problems that are related in spirit to the title problem. The variety is especially marked in Section 1, which concerns the role of reflections in illumination and billiards.

Problem 6 is conceptually the simplest. It does not involve distance or angle or area, only the incidence of points and lines. Problem 5 involves convex polygons as well, and Problem 3 involves areas of unions of disks, but these problems would also have been easily understood by the ancient Greeks. Problems 2 and 4 are simple, but they do involve more general convex sets than the other problems just mentioned. Problems 7 and 8 concern subdivisions of the entire plane, rather than just part of it. Problem 9 deals with a striking interaction between the simplest convex sets (circles and squares) and sets that probably no one would agree to call "geometric figures." Problem 10 involves some relationships between geometry and number theory, while Problems 11 and 12 involve topological notions.

Several of the problems are striking for the great disparity between what is known and what may be true. For some of them, this disparity can be expressed in a quantitative way. Problem 3 originates from the conjecture that when several congruent disks are pushed closer together, the area of their union can't increase. However, the best that has been proved is that it can't increase by a factor of more than 9. Problem 6 was motivated by a conjecture that when n points in the plane are not all collinear, and when all the lines determined by pairs of these points are added to the configuration, then one of the points is on at least $\frac{1}{2}n$ of the lines. Some small counterexamples show that $\frac{1}{2}n$ doesn't always work, but perhaps $\frac{1}{3}n$ does. It was proved initially (but not easily) that $10^{-1087}n$ always works, and that has been improved to $10^{-32}n$—still a long way from $\frac{1}{3}n$. A more technical result in Section 1 involves a smoothness assumption for the boundary curvature of a convex billiard table. The original theorem assumed the curvature to be 553 times differentiable. The 553 has been reduced to 6, but a further reduction is probably possible.

The most easily stated problems are 8 and 10. Problem 8 asks, "What is the minimum number of colors for painting the plane so that no two points at unit distance receive the same color?" It has been known since 1960, when the problem first appeared in print, that the number is 4, 5, 6, or 7, but no one has been able to narrow the possibilities further. Problem 10 asks, "Does every simple closed curve in the plane contain all four vertices of some square?" An affirmative answer has been known since 1913 for all sufficiently smooth convex curves, and more recent results have avoided the convexity assumption. However, there is a sense in which most simple closed curves are not smooth at all, and thus it is possible that the answer is negative for most curves. Despite their ease of statement (or perhaps because of it!), Problems 8 and 10 have, like many of the other problems considered here, been the subject of incorrect published results. As we mentioned in the Preface, it is tempting to believe that an easily stated problem should have an easy solution. The reader is cautioned against believing that for most of the problems stated here.

There are other good sources of information about easily understood unsolved geometric problems. We mention especially the book on tiling by B. Grünbaum and G. Shephard [GS4] and the books on lattice point problems by J. Hammer [Ham] and P. Erdős, P. Gruber, and Hammer [EGH]. In addition, there are three excellent collections of unsolved problems that will soon become books—those of Erdős and G. Purdy [EP3], of H. Croft, K. Falconer, and R. Guy [CFG], and of W. Moser and J. Pach [MP] (developed from an earlier collection of L. Moser). We should also mention the numerous articles of Erdős, Erdős and Purdy, L. Fejes Tóth, Grünbaum, and H. Hadwiger, some of which are included in our list of references. Another good source of geometric problems is the "Unsolved Problems" section in the *American Mathematical Monthly*. And there are many other sources—too many to mention them all. It seems clear that the oldest part of mathematics will continue to provide challenges (and surprises) for many years to come.

1. ILLUMINATING A POLYGON

Problem 1

Is each reflecting polygonal region illuminable?

Imagine life in a two-dimensional world where electricity and skilled construction are very expensive but reflecting materials are cheap. One might live in a room whose shape resembles that of Figure 1.1 and whose walls are mirrors, and one might hope to illuminate the entire room by a single light source. Can this always be done, regardless of the room's shape? If it can be done, how should one discover where to place the light source?

Light rays issue in all directions from the source, and each ray continues on its way according to the usual rule: "angle of reflection equals angle of incidence." The room is *illuminated* by a given source if each point of the room lies on at least one of the rays. For each position of the light source s, the points that are illuminated *directly* from s (without using reflections) form a set that is geometrically simple, for it's a union of straight-line segments issuing from s. (Such a set is said to be *star-shaped* from s). However, this simple picture rapidly becomes complicated as more and more reflections are taken into account, and when the number of reflections is