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ALGEBRA AND TRIGONOMETRY

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Preface

This text is written for the college student who has had at least a year of high school algebra and who has some familiarity with the basic notions of geometry. A knowledge of postulational geometry is not assumed, but some familiarity with the properties of real numbers (reviewed in Chapter 1) is expected.

The function concept is central in the text. Functions of real numbers are introduced early and considered throughout the rest of the book. Trigonometric functions are introduced initially as functions of real numbers, later as functions of angles or rotations. Some of the other functions considered are polynomial functions, logarithmic functions, exponential functions, and absolute value functions.

The book is one of few in which algebra and trigonometry are actually integrated and not merely placed side-by-side. This integration effects a significant economy of effort on the part of both student and instructor. For example, the student is asked to factor such expressions as $\sin^2 x - \cos^2 x$ in the same assignment as $x^2 - y^2$. Other features of the book that effect similar economies are: (1) Translations of functions are considered early and used not only in the usual manner but also in the development of trigonometric identities. (2) The basic theory of linear interpolation is considered once and then applied to tables of various kinds. (3) The concepts of relation and function are considered early (Chapter 2) and then applied to the circular functions and other functions encountered later.

It is often pointed out these days that the existence of computing machinery makes calculating skill less important than it once was, while the ability to innovate, to create new mathematics, and to formulate problems has become more important. While this is true, and the authors have placed strong emphasis on understanding by the student, they hold that it would be a serious mistake to de-emphasize calculating skills to a great extent. Hence, computational material has been included.

February 1967

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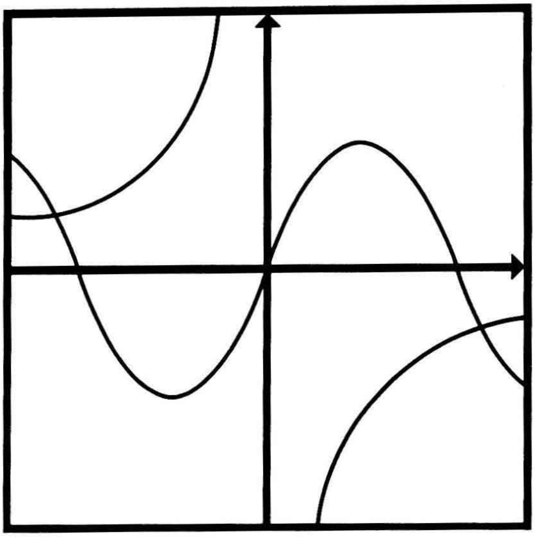
J.F.S.
A.M.

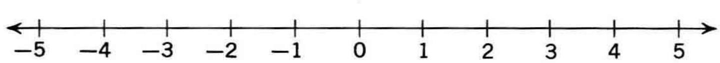
Contents



CHAPTER 1

The System of Real Numbers

Basic to algebra and trigonometry, as well as to most advanced mathematics, is the system of real numbers. You are already familiar with these numbers from your study of arithmetic and algebra. The system of real numbers contains one and only one number corresponding to each point of a line. The real numbers and the points of a line may be matched as shown below. The point which corresponds to the number 0



is called the *origin* of the number line. The positive numbers are then associated with the points on one half-line and the negative numbers with the points on the other half-line. This correspondence will help you visualize many ideas as we review the system of real numbers in Chapter 1.

1 · FUNDAMENTAL PROPERTIES OF REAL NUMBERS

The two fundamental operations in the system of real numbers are addition (+) and multiplication (·). We shall list some of the most basic and important properties of this system of numbers for purposes of review. The first five properties which follow involve the operation of addition.

Any two real numbers may be added, and furthermore, the sum will always be a unique real number. We say,

The set of real numbers is closed under addition.

The order in which any two numbers are added makes no difference. In other words, if a and b are any real numbers, then $a + b = b + a$. For example, $4.5 + \sqrt{2} = \sqrt{2} + 4.5$. In other words,

Addition is a commutative operation.

The manner in which real numbers are grouped for adding makes no difference. In other words, if a, b, and c are any real numbers, then $a + (b + c) = (a + b) + c$. For example, $3 + (7 + 9) = 3 + 16 = 19$, and also $(3 + 7) + 9 = 10 + 9 = 19$. We say,

Addition is an associative operation.

When 0 is added to any real number, the result is that same real number. In other words, if x is any real number, then $x + 0 = x$. We call 0 the additive identity:

For the set of real numbers, the number 0 is the additive identity.

If we choose any real number x, then there is some number which when added to it gives 0. We say that each number has an additive inverse:

Every real number has one and only one additive inverse.

We usually name the additive inverse of x either "^{-}x" or "$-x$."

For example,

If $x = 5$, then ^{-}x is negative five.
If $x = 0$, then ^{-}x is zero.
If $x = {}^{-}17$, then ^{-}x is seventeen.

We might also state this property as follows: If x is any real number, then there is a unique real number ^{-}x for which $x + {}^{-}x = 0$. From this property we also know that if $a = b$, then $^{-}a = {}^{-}b$.

The next five properties are similar to those we have reviewed. Those which follow involve the multiplication operation.

Any two real numbers may be multiplied, and the product is also a unique real number. In other words,

The set of real numbers is closed under multiplication.

For any real numbers a and b, $a \cdot b = b \cdot a$. So we say,

Multiplication is a commutative operation.

For any real numbers a, b, and c, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. So we say,

Multiplication is an associative operation.

When 1 is multiplied by any real number, the result is that same real number. In other words, if x is any real number, then $x \cdot 1 = x$. We call 1 the multiplicative identity:

For the set of real numbers, the number 1 is the multiplicative identity.

If we choose any real number x, other than zero, then there is some number which when multiplied by it gives 1. We say that each number has a multiplicative inverse:

Every non-zero real number has one and only one multiplicative inverse. Zero has no multiplicative inverse.

We often use the symbol "$\frac{1}{x}$" to represent the multiplicative inverse of x. Multiplicative inverses are also called *reciprocals*.

For example,

The multiplicative inverse of 3 is $\frac{1}{3}$.

The reciprocal of $^{-}6$ is $\frac{1}{^{-}6}$.

The reciprocal of $\frac{1}{\sqrt{3}}$ is $\sqrt{3}$.

We might also state the property of reciprocals as follows: If x is any non-zero real number, then there is a real number $\frac{1}{x}$ for which $x \cdot \frac{1}{x} = 1$.

The final property which we list in this section involves both multiplication and addition and states that if a number is to be multiplied by a sum, we may multiply it by the addends individually and then add, without changing the result. For example,

$$3 \cdot (4 + 7) = 3 \cdot 11 = 33.$$
$$(3 \cdot 4) + (3 \cdot 7) = 12 + 21 = 33.$$

Hence we say that,

Multiplication is distributive over addition.

We might also state this property as follows: If x, y, and z are any real numbers, then $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

EXERCISES · SET 1

1. Tell which property or properties of real numbers is illustrated by each of the following:

a. $59 + 34 = 34 + 59$
b. $16 \cdot (3 + 11) = (16 \cdot 3) + (16 \cdot 11)$
c. $^{-}4 + (3 + 1.2) = (^{-}4 + 3) + 1.2$
d. $^{-}4 + (3 + 1.2) = (3 + 1.2) + (^{-}4)$
e. $16 \cdot (5 + 7) = (5 + 7) \cdot 16$
f. $4 \cdot (9 + 4.3) = (9 + 4.3) \cdot 4$
g. $6 \cdot (4 \cdot 7) = (6 \cdot 4) \cdot 7$
h. $(4 \cdot 9) + (5 \cdot 9) = 9 \cdot 9$
i. $4 + (3 + 5) = (5 + 4) + 3$
j. $0 + 4.5 = 4.5$
k. $1 \cdot 13 = 13$
l. $(5 - 4) \cdot 12 = 12$
m. $2x + 2y = 2(x + y)$
n. $9y = 4y + 5y$
o. $2x(4 + y) = 8x + 2xy$
p. $3x + (3y - 3y) = 3x$

2. Let m and n represent any real numbers.

a. Is the sum $m + n$ necessarily a real number? Explain.
b. Is the product $m \cdot n$ necessarily a real number? Explain.
c. Is the sum $m + {}^{-}n$ necessarily a real number? Explain.
d. Is the product $m \cdot \frac{1}{n}$ necessarily a real number? Explain.

3. **a.** Is $(20 - 12) - 3 = 20 - (12 - 3)$? Is subtraction of real numbers associative?
b. Is $(16 \div 8) \div 2 = 16 \div (8 \div 2)$? Is division of real numbers associative?

4. If addition were distributive over multiplication, then for any real numbers a, b, and c, it would be true that $a + (b \cdot c) = (a + b) \cdot (a + c)$. Is this true for all real numbers?

5. Add and simplify:

a. $3 + {}^{-}1$
b. $17 + {}^{-}5$
c. ${}^{-}3 + 7$
d. ${}^{-}5 + 15$
e. ${}^{-}2 + 1$
f. ${}^{-}16 + 4$
g. $6 + {}^{-}10$
h. $12 + {}^{-}20$
i. ${}^{-}2 + {}^{-}3$
j. ${}^{-}14 + {}^{-}5$
k. ${}^{-}7 + {}^{-}5$

6. Apply the distributive property. For example, $5(x + y) = 5 \cdot x + 5 \cdot y$.

a. $6(x + y)$
b. $7(x + z)$
c. $x(4 + y)$
d. $2x(y + z)$
e. ${}^{-}5(x + y)$
f. ${}^{-}3y(2 + z)$

7. Factor. For example, $3x + 3y = 3(x + y)$.

a. $5x + 5y$
b. $3x + 3z$
c. $3x + 5x$
d. $xy + 3y$
e. ${}^{-}2x + xy$
f. ${}^{-}4x + {}^{-}yx$

2 · SYMBOLISM

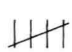
In mathematics the careful use of symbols is of utmost importance. It is important to understand the meanings of various kinds of symbols in order that we may use them correctly.

Numerals are symbols we write to represent numbers. There are many kinds of numerals. Hindu-Arabic numerals are most familiar because we use them daily. Roman numerals are less familiar, but are known to all of us. You have probably used another kind of numeral, or symbol for numbers, in tallying votes for a class election. In other words, a numeral is a symbol for a number, or a "name" of a number.

It is important to remember that numbers and numerals are different. They are different in the same way that you and your name are different. If you write the name "John" on the chalkboard, then it is clear that you did not write a person, but his name. In the same way, if you write the numeral "2" you have not written a number, but a name, or numeral, which represents the number two. You may say you have written the number two, but this is not a precise way of speaking.

Remember that a number has many numerals or names. For example, some of the symbols which name the number five are:

$5 \qquad V \qquad 1 + 4 \qquad 10 - 5$

$10 \div 2$  $1 + 1 + 1 + 1 + 1$

Variables and Constants

Variables are symbols which hold a place in an expression or sentence, and for which names of any of the members of a specified set may be substituted. This set is called the *replacement set.* A numeral or any symbol which holds a place for a particular number is called a *constant.* The most usual kind of symbol for a variable is a letter. For example, in the expression $y + 2$, the letter y is a variable. In this case, it holds a place in which various numerals may be substituted. In the sentence $b - \pi = c + 2$, the letters b and c are variables, whereas π is a constant. Usually the context will make it clear whether a letter is a variable or a constant.

The set of symbols which hold a place for the variable in an expression or sentence is called the replacement set.

Equality

The symbol "=" means "is equal to" and is used to construct a sentence. Its role in the sentence is that of a verb. Any sentence which has this symbol for its principal verb is called an *equation.*

It is important to remember that sentences can be false. The fact that a sentence is false in no way keeps it from being a well-constructed sentence expressing an idea. For example, the sentence "The temperature in Miami is $^{-}75^\circ$." is false, but the idea it expresses is clear, and it definitely is a sentence. In the same way the sentence "$3 + 5 = 6$" is false, but it is still a sentence.

One may wonder just what meaning is expressed by a sentence which has = for its verb. The simplest way to give a definition for the meaning of such a sentence is as follows:

A sentence with = for its verb says that the symbol before and the symbol after the symbol = represent the same thing.

It should be obvious from this definition that any equation may be reversed, or the symbols on the right and left of the equals symbol may be interchanged. You can see that "$x - 5 = 3$" and "$3 = x - 5$" say exactly the same thing. If an equation contains one or more variables it may be neither true nor false. Such a sentence is called an *open sentence.* For example, the sentence "$x + 5 = 7$" is neither true nor false. The variable x is merely holding a place for the substitution of numerals. If we replace x by 7 then the equation becomes a false sentence; substituting 2 for x renders it true. As it stands, it is neither true nor false.

Algebraic Expressions

Much of the symbolism which we need in algebra and trigonometry is familiar, but before continuing, it will be useful to review the meanings of the common symbols to make sure that they are clear. Of primary interest is the class of symbols known as *algebraic expressions.* An algebraic expression may consist of a single numeral or a single variable; or it may consist of various combinations of numerals and variables, together with the usual operation signs, such as $+$, $-$, $\cdot$, $\div$, and so on, as well as such symbols as parentheses and brackets. Here are some examples of algebraic expressions:

$$2x + y \qquad x^2 - 3y^m \qquad \sqrt{x} - 5$$

$$4 \qquad 3ay^2 - 2(dy + 5a) \qquad \frac{3x + 7}{a - b}$$

Exponents

Using exponents enables us to eliminate much writing. For example, the expression $t \cdot t \cdot t \cdot t \cdot t \cdot t \cdot t$ takes more effort to write and is harder to read than its equivalent, t^7. We see that there are seven factors t in the first expression. The 7 in the second expression, written as a superscript, indicates a product of 7 factors, each of which is t. The number t is sometimes called the *base.* In general, the exponent symbol, when *and only when* it represents a natural number greater than 1, means that the base occurs that many times as a factor in the product. An exponent 1, as in b^1, effects no change in the meaning of an expression. In other words, our definition of the meaning of exponents is

$y^n = y \cdot y \cdot y \cdot y \cdots y$ (n factors), if and only if n represents a natural number greater than 1.
Also, $\qquad y^1 = y.$

Symbols of Grouping

Symbols of grouping are used in algebraic expressions to indicate that the symbols enclosed represent a single number. For example, in the expression $3t - (4c + 7)$, the expression $4c + 7$ represents a single number, which is to be subtracted from $3t$. Parentheses are the most common symbols of grouping, but *brackets,* [ ], and *braces,* { }, are also used in the same way.

Symbols of grouping are often omitted *by agreement.* For example, in $4c + 7$ we understand that the multiplication of 4 by c is to be performed before the addition of 7. Thus the expression means $(4c) + 7$, and *not* $4(c + 7)$. When several operations are indicated we agree that multiplication and division are performed before addition and subtraction.

EXERCISES · SET 2

1. **a.** Write ten different numerals for the number twelve.
b. Write ten different numerals for the number twenty-six.

2. In each of the following sentences, which symbols are variables? Which are constants?

a. $C = 2\pi r$
b. $x + 6 = 7$
c. $A = \frac{1}{2}bh$
d. $A = \frac{1}{2}h(b + b_1)$

3. Multiply and simplify. (For example, $3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3) = 3^7$.)

a. $4^5 \cdot 4^2$
b. $5^2 \cdot 5^6$
c. $6^3 \cdot 6^7$
d. $10^8 \cdot 10^5$
e. $x^2 \cdot x^7$
f. $(4x)^3 \cdot (4x)^4$
g. $6y^4 \cdot 3y^5$
h. $(^-3)^5 \cdot (^-3)^2$
i. $5x^5 \cdot 3x^5$
j. $10x^4 \cdot 12y^5$

4. Divide and simplify:

Examples:

$$\frac{3^5}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3} = 3 \cdot 3 \cdot 3 = 3^3$$

$$\frac{5^3}{5^7} = \frac{5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5} = \frac{1}{5 \cdot 5 \cdot 5 \cdot 5} = \frac{1}{5^4}$$

a. $\frac{4^5}{4^2}$
b. $\frac{3^8}{3^6}$
c. $\frac{(^-4)^5}{(^-4)^3}$
d. $\frac{3x^6}{x^3}$
e. $\frac{10^{15}}{10^3}$
f. $\frac{3^6}{3^8}$
g. $\frac{(^-5)^8}{(^-5)^{12}}$
h. $\frac{x^4}{x^{10}}$
i. $\frac{3y^{25}}{3y^{25}}$
j. $\frac{7x^5}{5x^8}$

ALGEBRAIC SYMBOLISM

The symbols used in algebra today were not all invented at the same time, but contributions to the symbolism have been made by a number of mathematicians at different times in history.

Diophantus of Alexandria (about 200 A.D.) was probably the first mathematician to use letters to represent numbers. François Viète, of France (1540–1603), used vowels for unknowns, and his countryman, Descartes (about 1630), originated the use of *x, y, z* for variables and *a, b, c* for constants.

The + and − signs probably first appeared in a book written by Widmann, a German, in 1489. The cross as a multiplication symbol was used in 1631, in England, by William Oughtred, and the raised dot was used about the same time by the German, Leibnitz. The division sign, ÷, was first used in print in 1668 in an algebra book by the Swiss mathematician, Johann Rahn. The Arabs used symbols like *a/b* to represent division as early as 1000 A.D.

Several symbols for equality have been used. The present one, =, was introduced by the Englishman, Robert Recorde, in an algebra text, *The Whetstone of Witte,* in 1557. He defends his choice of the pair of horizontal line segments by saying "noe 2 thynges can be moare equalle."

3 · SETS AND SENTENCES

Let us briefly review set notation. In general, we denote or name a set using braces. Here is an example,

$$\{5, 6, 2, 7, 11\}.$$

This is a name for the set which contains the numbers 5, 6, 2, 7, and 11. When the members of a set are too numerous to mention, we use an adapted notation,

$$\{x \mid x < 5\}.$$

This is read "the set of all x such that x is less than 5."

The *intersection* of sets is indicated by using the symbol $\cap$. The intersection of several sets consists of the elements common to those sets. For example,

$$\{3, 4, 5, 8, 10\} \cap \{1, 2, 3, 4, 5, 6, 7\} = \{3, 4, 5\}.$$

The *union* of sets is indicated by using the symbol $\cup$, and is the set obtained by combining the sets. For example,

$$\{0, 3, 5, 9\} \cup \{0, 1, 2, 3, 4\} = \{0, 1, 2, 3, 4, 5, 9\}.$$

A set with no elements is called the *empty set,* ϕ.