

PURCELL



3rd edition

Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632

ulus with geometry

The bottom of the page features three thick, dark horizontal bars stacked vertically, serving as a decorative footer.

Library of Congress Cataloging in Publication Data

PURCELL, EDWIN JOSEPH, (date)
Calculus with analytic geometry.

Includes index.

1. Calculus. 2. Geometry, Analytic. I. Title.
QA303.P99 1978 515'.15 77-7977
ISBN 0-13-112052-2

CALCULUS WITH ANALYTIC GEOMETRY, 3rd edition *Edwin J. Purcell*

Cover Illustration: "Color Ribbon" by Joanna Pinsky

© 1978, 1972, 1965 by Prentice-Hall, Inc., Englewood Cliffs, N.J. 07632

All rights reserved. No part of this book
may be reproduced in any form or
by any means without permission in writing
from the publisher.

Printed in the United States of America

10 9 8 7 6 5 4

Prentice-Hall International, Inc., *London*

Prentice-Hall of Australia Pty. Limited, *Sydney*

Prentice-Hall of Canada, Ltd., *Toronto*

Prentice-Hall of India Private Limited, *New Delhi*

Prentice-Hall of Japan, Inc., *Tokyo*

Prentice-Hall of Southeast Asia Pte. Ltd., *Singapore*

Whitehall Books Limited, *Wellington, New Zealand*

preface

This book presents a first course in calculus and analytic geometry. The author has tried to write it in a simple and straightforward style, with ample explanations, an abundance of illustrative examples, and carefully graded exercise sets, so that it will be unusually suitable for a reader of average ability to study alone or with minimum help from a teacher. There is enough material for three semesters.

Starting with simple first principles, each new concept is motivated by a natural, intuitive introduction. Seven basic concepts are stressed: function, limit of a function, continuity, derivative, antiderivative, definite integral, and infinite series. Effort is made to impress on the reader that a mastery of these ideas is indispensable in acquiring a genuine understanding of calculus.

At the same time, there is an abundance of material dealing with the degree of accuracy of computed results and with other aspects of the computational work that is so important for progress in science and technology.

Since ϵ , δ methods are necessary for a proper definition of the limit of a function, a very thorough treatment of inequalities and absolute values precedes it. However, the use of ϵ , δ methods is minimized in later work by utilizing limit theorems wherever possible.

Set notation is introduced early. It is employed when clearly advantageous, but not slavishly.

Vectors in two- and three-dimensional space are presented with a firm mathematical basis and are applied widely. Vectors do not supplant a sound foundation in Cartesian plane analytic geometry but complement it and make possible a more concise formulation of some of the theorems that were first derived in the classical

Cartesian manner. In three-dimensional analytic geometry, vectors are used from the outset. This avoids duplication of effort and contributes to a better understanding of both subjects.

To make this third edition more readable for the increasing numbers of students with less preparation for calculus than formerly, the following improvements have been made.

Many new illustrative examples with complete solutions clarify the theorems, definitions, and techniques. Students who study them should be able to do most of the exercises.

The exercise sets have been reworked and excessive algebraic manipulation has been eliminated. Each set now starts with easy variations of the illustrative examples, progresses through exercises of increasing difficulty, and concludes with some to challenge the stronger students.

Each chapter begins with an intuitive preview of the main ideas to be discussed and their relation to what has gone before. This helps the reader to see the developing calculus as a whole rather than as a series of isolated processes.

At the end of each chapter there is a set of review exercises. A student often finds it easy to work the exercises in a particular section because the method has just been explained. But a set of miscellaneous review exercises based on the material of an entire chapter causes the student to review the chapter and gain a better understanding of it.

Many proofs have been simplified and some of the more tedious ones have been moved to the appendix.

Stronger students will find that the logical development of this third edition and the careful statement of its theorems maintain the integrity of earlier editions. The organization has been improved by moving infinite series forward to Chapter 14, so that the first fifteen chapters now constitute a two semester course in single variable calculus. Chapters 16 to 19 treat the calculus of two or more variables.

The preliminary material has been shortened in order to start the actual calculus sooner. The chapter on the definite integral has been rewritten; it is simpler, more direct, and easier to understand. The treatment of trigonometric functions has been much improved.

There are new sections on Lagrange multipliers and on surface area.

Throughout this book the principal definitions and theorems are prominently labeled, numbered, and displayed, both for easy reference and to keep the main structure of the material before the reader's eyes. The number 7.3.4, for example, refers to the fourth numbered definition or theorem in Section 3 of Chapter 7. The number 14.6 refers to Section 6 of Chapter 14. Fig. 11-5 indicates the fifth figure in Chapter 11.

The most used theorems and definitions are printed in color to enable the student to concentrate his efforts to advantage.

I wish to thank many users of the earlier editions, both faculty and students, for their comments, criticism and encouragement. My thanks are also due to my wife, Bernice Lee Purcell, who made the index and typed the manuscript.

EDWIN J. PURCELL
University of Arizona

contents

preface

xiii

preliminaries

1

- 1.1 INTRODUCTION 1
- 1.2 SETS 4
- 1.3 REAL NUMBERS 7
- 1.4 THE COORDINATE LINE. INTERVALS 12
- 1.5 INEQUALITIES 16
- 1.6 ABSOLUTE VALUE 21
- 1.7 INDUCTION 27
- 1.8 REVIEW EXERCISES 29



the cartesian plane. functions

31

- 2.1 RECTANGULAR CARTESIAN COORDINATES 31
- 2.2 SLOPE AND MIDPOINT OF A LINE SEGMENT 35
- 2.3 SUBSETS OF THE CARTESIAN PLANE 38
- 2.4 THE STRAIGHT LINE 41
- 2.5 PARALLEL AND PERPENDICULAR LINES. CIRCLES 45
- 2.6 SKETCHING GRAPHS OF EQUATIONS 51
- 2.7 GRAPHS OF INEQUALITIES 55
- 2.8 FUNCTIONS 60
- 2.9 SPECIAL FUNCTIONS 66
- 2.10 OPERATIONS ON FUNCTIONS 69
- 2.11 INVERSE OF A FUNCTION 72
- 2.12 REVIEW EXERCISES 76

v

3

limits and continuity

77

- 3.1 INTRODUCTION TO LIMITS 77
- 3.2 DEFINITION OF THE LIMIT OF A FUNCTION 83
- 3.3 THEOREMS ON LIMITS 91
- 3.4 ONE-SIDED LIMITS 98
- 3.5 CONTINUITY 101
- 3.6 INCREMENTS 107
- 3.7 REVIEW EXERCISES 110

4

the derivative

112

- 4.1 TANGENT LINE TO A CURVE 112
- 4.2 INSTANTANEOUS VELOCITY 117
- 4.3 THE DERIVATIVE 121
- 4.4 APPLICATIONS OF THE DERIVATIVE. RATE OF CHANGE 125
- 4.5 THE DERIVATIVE AND CONTINUITY 129
- 4.6 REVIEW EXERCISES 133

5

formulas for differentiation of algebraic functions. applications

135

- 5.1 DERIVATIVE OF A POLYNOMIAL FUNCTION 136
- 5.2 DERIVATIVE OF A PRODUCT OR QUOTIENT OF FUNCTIONS 139
- 5.3 CHAIN RULE FOR DIFFERENTIATING COMPOSITE FUNCTIONS 142
- 5.4 DERIVATIVE OF ANY RATIONAL POWER OF A FUNCTION 147
- 5.5 DERIVATIVES OF HIGHER ORDER 150
- 5.6 IMPLICIT DIFFERENTIATION 153
- 5.7 TANGENTS AND NORMALS 156
- 5.8 RELATED RATES 159
- 5.9 DIFFERENTIALS 164
- 5.10 DIFFERENTIALS AS LINEAR APPROXIMATIONS 167
- 5.11 DERIVATIVE OF THE INVERSE OF A FUNCTION 171
- 5.12 REVIEW EXERCISES 173

6

further applications of the derivative

175

- 6.1 ROLLE'S THEOREM 176
- 6.2 THE MEAN VALUE THEOREM 181
- 6.3 INCREASING FUNCTIONS AND DECREASING FUNCTIONS 185
- 6.4 VELOCITY AND ACCELERATION IN RECTILINEAR MOTION 190

- 6.5 LOCAL MAXIMA AND MINIMA 192
- 6.6 SECOND DERIVATIVE TEST FOR LOCAL EXTREME VALUES 197
- 6.7 FINDING EXTREME VALUES 198
- 6.8 APPLIED PROBLEMS IN MAXIMA AND MINIMA 201
- 6.9 APPLICATIONS TO ECONOMICS 207
- 6.10 CONCAVITY, POINTS OF INFLECTION 210
- 6.11 LIMITS AS $x \rightarrow \infty$. ASYMPTOTES 215
- 6.12 GRAPH SKETCHING 221
- 6.13 NEWTON'S METHOD FOR DETERMINING THE ROOTS
OF $f(x) = 0$ 225
- 6.14 REVIEW EXERCISES 229

7

antiderivatives

232

- 7.1 ANTIDERIVATIVES 232
- 7.2 FINDING ANTIDERIVATIVES 234
- 7.3 GENERALIZED POWER FORMULA FOR ANTIDERIVATIVES 237
- 7.4 SOME APPLICATIONS OF ANTIDERIVATIVES.
DIFFERENTIAL EQUATIONS 240

8

the definite integral

243

- 8.1 INTRODUCTION TO AREA 243
- 8.2 THE SUMMATION NOTATION 251
- 8.3 THE DEFINITE INTEGRAL 254
- 8.4 THE FUNDAMENTAL THEOREM OF CALCULUS 262
- 8.5 PROPERTIES OF DEFINITE INTEGRALS 268
- 8.6 THE MEAN VALUE THEOREM FOR INTEGRALS 272
- 8.7 APPROXIMATE INTEGRATION BY THE TRAPEZOIDAL RULE 274
- 8.8 REVIEW EXERCISES 278

9

applications of the definite integral

279

- 9.1 PLANE AREAS 279
- 9.2 VOLUME OF A SOLID OF REVOLUTION 286
- 9.3 VOLUME BY CYLINDRICAL SHELLS 292
- 9.4 WORK 296
- 9.5 LIQUID PRESSURE 299
- 9.6 ARC LENGTH 302
- 9.7 AREA OF A SURFACE OF REVOLUTION 306
- 9.8 CENTROID OF A PLANE REGION 309
- 9.9 CENTROID OF A SOLID OF REVOLUTION 320
- 9.10 APPLICATIONS TO ECONOMICS 326
- 9.11 REVIEW EXERCISES 329



transcendental functions

331

- 10.1 THE NATURAL LOGARITHM FUNCTION 332
- 10.2 INVERSE OF A FUNCTION 340
- 10.3 THE EXPONENTIAL FUNCTION 343
- 10.4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS
WITH BASES OTHER THAN e 348
- 10.5 TRIGONOMETRIC FUNCTIONS 353
- 10.6 SOME TRIGONOMETRIC LIMITS 360
- 10.7 DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS 363
- 10.8 INVERSE TRIGONOMETRIC FUNCTIONS 371
- 10.9 HYPERBOLIC FUNCTIONS 377
- 10.10 INVERSE HYPERBOLIC FUNCTIONS 382
- 10.11 REVIEW EXERCISES 384



techniques of integration

386

- 11.1 INTRODUCTION 386
- 11.2 THE BASIC INTEGRATION FORMULAS 388
- 11.3 INTEGRATION BY SUBSTITUTION 392
- 11.4 SUBSTITUTION IN THE REMAINING BASIC FORMULAS
FOR INTEGRATION 397
- 11.5 INTEGRATION BY PARTS 401
- 11.6 SOME TRIGONOMETRIC INTEGRALS 405
- 11.7 RATIONALIZING SUBSTITUTIONS 411
- 11.8 INTEGRANDS INVOLVING $Ax^2 + Bx + C$ 417
- 11.9 DEFINITE INTEGRALS. CHANGE OF LIMITS 421
- 11.10 INTEGRATION OF RATIONAL FUNCTIONS
BY PARTIAL FRACTIONS 424
- 11.11 RATIONAL FUNCTIONS OF $\sin x$ AND $\cos x$ 430
- 11.12 REVIEW EXERCISES FOR INTEGRATION 432
- 11.13 TABLES OF INTEGRALS 434
- 11.14 SEPARABLE DIFFERENTIAL EQUATIONS. APPLICATIONS 434
- 11.15 SIMPSON'S RULE 442



conics. polar coordinates

448

- 12.1 CONIC SECTIONS 448
- 12.2 DEFINITION OF A CONIC 450
- 12.3 THE PARABOLA ($e = 1$) 453

- 12.4 CENTRAL CONICS ($e \neq 1$) 457
- 12.5 THE ELLIPSE ($e < 1$) 459
- 12.6 THE HYPERBOLA ($e > 1$) 462
- 12.7 OTHER DEFINITIONS OF CENTRAL CONICS 466
- 12.8 TRANSLATION OF COORDINATE AXES 468
- 12.9 THE ROTATION TRANSFORMATION 474
- 12.10 THE GRAPH OF ANY SECOND-DEGREE EQUATION 480
- 12.11 POLAR COORDINATES 485
- 12.12 THE GRAPH OF A POLAR EQUATION 491
- 12.13 LINES, CIRCLES, AND CONICS 501
- 12.14 ANGLE FROM THE RADIUS VECTOR TO THE TANGENT LINE 506
- 12.15 INTERSECTION OF CURVES IN POLAR COORDINATES 510
- 12.16 PLANE AREAS IN POLAR COORDINATES 513
- 12.17 REVIEW EXERCISES 519

13

indeterminate forms. improper integrals

521

- 13.1 CAUCHY'S MEAN VALUE THEOREM 522
- 13.2 INDETERMINATE FORMS 523
- 13.3 L'HÔPITAL'S RULES 524
- 13.4 OTHER INDETERMINATE FORMS 528
- 13.5 INFINITE LIMITS OF INTEGRATION 530
- 13.6 BOUNDED SETS. THE COMPLETENESS AXIOM 535
- 13.7 COMPARISON TEST FOR CONVERGENCE 536
- 13.8 INFINITE INTEGRANDS 540
- 13.9 REVIEW EXERCISES 545

14

infinite series

547

- 14.1 SEQUENCES 548
- 14.2 INFINITE SERIES 552
- 14.3 TESTS FOR CONVERGENCE OF SERIES OF POSITIVE TERMS 559
- 14.4 ALTERNATING SERIES. ABSOLUTE CONVERGENCE 566
- 14.5 POWER SERIES 571
- 14.6 FUNCTIONS DEFINED BY POWER SERIES 575
- 14.7 TAYLOR'S THEOREM 579
- 14.8 OTHER FORMS OF THE REMAINDER
IN TAYLOR'S THEOREM 584
- 14.9 REVIEW EXERCISES 591

parametric equations and vectors in the plane

594

- 15.1 PLANE CURVES 594
- 15.2 FUNCTIONS DEFINED BY PARAMETRIC EQUATIONS 601
- 15.3 LENGTH OF A PLANE ARC 605
- 15.4 VECTORS IN THE PLANE 610
- 15.5 SCALARS, DOT PRODUCT, AND BASIS VECTORS 615
- 15.6 VECTOR FUNCTIONS 621
- 15.7 CURVILINEAR MOTION 624
- 15.8 CURVATURE 628
- 15.9 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION 632
- 15.10 REVIEW EXERCISES 635

three-dimensional spaces

637

- 16.1 CARTESIAN COORDINATES IN THREE-SPACE 637
- 16.2 THREE-DIMENSIONAL VECTORS 643
- 16.3 THE CROSS PRODUCT 651
- 16.4 PLANES IN THREE-DIMENSIONAL SPACE 655
- 16.5 LINES IN THREE-SPACE 663
- 16.6 SURFACES 669
- 16.7 CYLINDERS. SURFACES OF REVOLUTION 670
- 16.8 SYMMETRY, TRACES, AND PLANE SECTIONS OF A SURFACE 675
- 16.9 QUADRIC SURFACES 679
- 16.10 PROCEDURE FOR SKETCHING A SURFACE 683
- 16.11 REVIEW EXERCISES 686

vector functions in three-dimensional space

688

- 17.1 VECTOR FUNCTIONS 688
- 17.2 VELOCITY, ACCELERATION, AND ARC LENGTH 693
- 17.3 CURVATURE, VECTOR COMPONENTS 697
- 17.4 THE LAWS OF PLANETARY MOTION 702
- 17.5 REVIEW EXERCISES 706

functions of two or more variables

707

- 18.1 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE 707
- 18.2 PARTIAL DERIVATIVES 712
- 18.3 LIMITS AND CONTINUITY 717
- 18.4 INCREMENTS AND DIFFERENTIALS 723
- 18.5 CHAIN RULE 726

- 18.6 THE DIRECTIONAL DERIVATIVE AND THE GRADIENT 730
- 18.7 TANGENT PLANES TO A SURFACE. EXTREMA OF A FUNCTION OF TWO VARIABLES 739
- 18.8 CONSTRAINED MAXIMA AND MINIMA. LAGRANGE MULTIPLIERS 746
- 18.9 EXACT DIFFERENTIALS 750
- 18.10 LINE INTEGRALS 753
- 18.11 WORK 760
- 18.12 DIVERGENCE AND CURL 765
- 18.13 REVIEW EXERCISES 767

19

multiple integrals

769

- 19.1 DOUBLE INTEGRALS 769
- 19.2 EVALUATION OF DOUBLE INTEGRALS BY ITERATED INTEGRALS 775
- 19.3 OTHER APPLICATIONS OF DOUBLE INTEGRALS 782
- 19.4 GREEN'S THEOREM 788
- 19.5 DOUBLE INTEGRALS IN POLAR COORDINATES 794
- 19.6 SURFACE AREA 800
- 19.7 TRIPLE INTEGRALS 804
- 19.8 APPLICATIONS OF TRIPLE INTEGRALS 810
- 19.9 CYLINDRICAL COORDINATES 813
- 19.10 SPHERICAL COORDINATES 819

20

differential equations

823

- 20.1 INTRODUCTION 823
- 20.2 HOMOGENEOUS EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE 826
- 20.3 EXACT EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE 828
- 20.4 LINEAR EQUATIONS OF THE FIRST ORDER 830
- 20.5 SECOND-ORDER EQUATIONS SOLVABLE BY FIRST-ORDER METHODS 832
- 20.6 LINEAR EQUATIONS OF ANY ORDER 837
- 20.7 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS 837
- 20.8 SOLUTION OF SECOND-ORDER HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS 839
- 20.9 NONHOMOGENEOUS LINEAR EQUATIONS OF ORDER TWO 842
- 20.10 A VIBRATING SPRING 846
- 20.11 ELECTRIC CIRCUITS 850
- 20.12 REVIEW EXERCISES 855

appendix

857

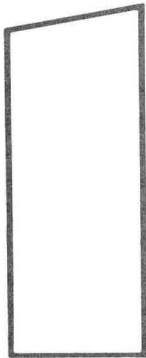
- A.1 THEOREMS ON LIMITS 857
- A.2 THEOREMS ON CONTINUOUS FUNCTIONS 861
- A.3 THE CHAIN RULE 864
- A.4 PROPERTIES OF DEFINITE INTEGRALS 866
- A.5 SOME PROPERTIES OF DETERMINANTS 870
- A.6 FORMULAS FROM GEOMETRY AND TRIGONOMETRY 878
- A.7 A SHORT TABLE OF INTEGRALS 879
- A.8 NUMERICAL TABLES 887

answers to odd-numbered exercises

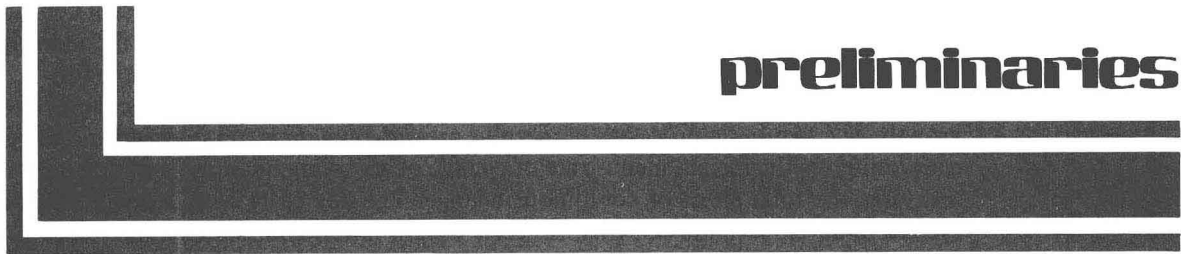
900

index

939



preliminaries



For students who are familiar with the preliminary concepts discussed in this chapter, a careful reading of most of it will suffice. However, *inequalities* and *absolute value* are so important in calculus that a mastery of Sections 1.5 and 1.6 is indispensable for success in studying this book. These topics are seldom covered adequately in high school, and a large proportion of the exercises in Sections 1.5 and 1.6 should be worked.

1.1 INTRODUCTION

Prior to the seventeenth century, algebra and geometry were studied as separate, unconnected subjects. The Greeks had perfected elementary geometry two thousand years ago, and in the centuries that followed the Hindus and Arabs cultivated algebra. Their algebra dealt with numbers, whereas Euclidian geometry was concerned with points, lines, planes, and the like.

There seemed little connection between algebra and geometry until the seventeenth century when two French mathematicians, René Descartes (1596–1650), who was also a philosopher, and Pierre de Fermat (1601–1655), invented a method, now called *analytic geometry*, that uses algebraic operations and equations to solve geometric problems; their method also shed new light on algebra by exhibiting its equations as geometric curves.

The basis for analytic geometry was Descartes' coordinate system, which associated the numbers of algebra with the points of geometry. By means of Cartesian

coordinates, large parts of algebra and geometry were seen to be two aspects of the same thing, somewhat as two different languages may express the same meaning. For instance the algebraic statement “Two distinct equations of the first degree in two variables have a single common solution or none” is equivalent to the geometric theorem “Two distinct lines in the same plane intersect in a single point or are parallel.”

The names generally associated with the invention of *calculus* are Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). Newton, an Englishman, developed calculus as a tool for his investigations in physics and astronomy. The German, Leibniz, was a universal genius who, independently of Newton and almost simultaneously, also developed calculus.

Calculus is based on the properties of numbers, and by using a Cartesian coordinate system, much of calculus can be presented in geometric terms. Thus the recently discovered analytic geometry was an ideal prelude to the invention of calculus.

Calculus, unlike the mathematics that preceded it, is the study of change and growth. The two basic processes of calculus are *differentiation* and *integration*. Differentiation gives the instantaneous rate of change of a varying quantity, and integration measures the total effect of continuous change. The key to Newton’s and Leibniz’s success in developing the calculus was their insight into the intimate relation between differentiation and integration as inverse processes, somewhat as multiplication and division of numbers are inverse operations.

Many scattered ideas from calculus were known to predecessors of Newton and Leibniz, even as far back as Archimedes (287–212 B.C.), who, without any algebra, succeeded in finding the areas of circles and regions under a parabola. For circles, he computed the areas of inscribed regular polygons of more and more sides. As the number of sides increased, the areas of the polygons increased and approached the area of the circle as a limit (Fig. 1-1). This is an example of integration.

The area of a circle is the limit of the area of an inscribed regular polygon of n sides as the number of sides, n , increases indefinitely

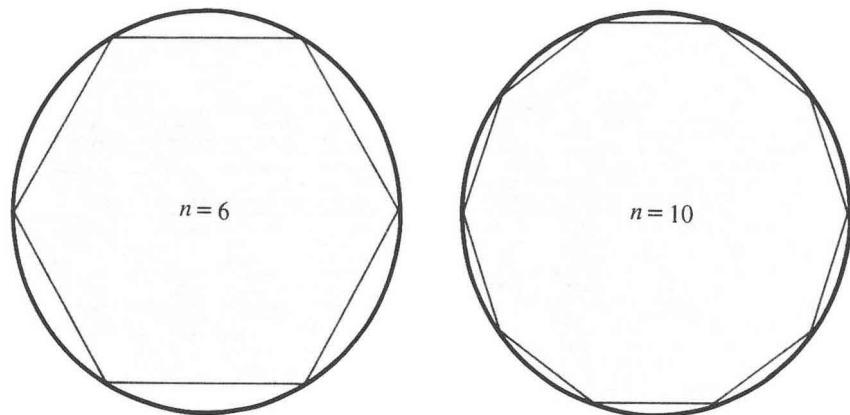


Figure 1-1

In the generation just before Newton, the problem of finding maximum and minimum values of a function was solved for some individual cases by finding the points on its graph where the tangent line is horizontal (Fig. 1-2). This led to a method for determining the direction of the tangent line to a curve at any point on the curve.

Let P be an arbitrarily chosen point on a curve and draw the secant line through P and a neighboring point Q on the curve (Fig. 1-3). Draw the vertical and horizontal

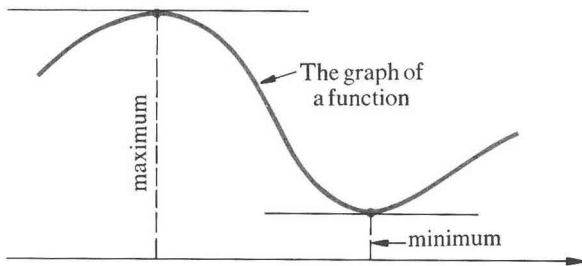


Figure 1-2

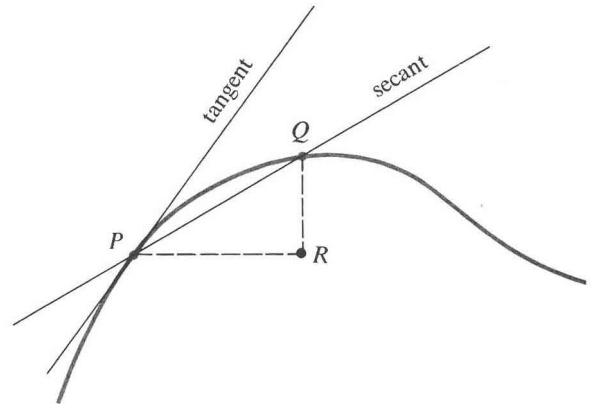


Figure 1-3

line segments, RQ and PR ; the ratio of their lengths, RQ/PR , is a measure of the steepness of ascent of a point on the secant and hence of the direction of the secant. Keeping P fixed on the curve, allow Q to approach P along the curve. This causes the secant PQ to rotate about P and approach the position of the tangent line at P . The limit of the ratio RQ/PR as Q approaches P along the curve gives the direction of the tangent line at P . This is an example in differentiation.

Notice that in both examples the word *limit* was used. Limit is the most important concept in calculus and is what distinguishes calculus from all previous mathematics.

Thus Newton and Leibniz were not the first to differentiate or integrate. In particular, Isaac Barrow, Newton's teacher at Cambridge, understood the area problem and the tangent problem and probably knew that they were inverse to each other. The importance of Newton and Leibniz in calculus resulted from their consolidation of the known fragments into a general method, incorporating what is now known as the fundamental theorem, which is applicable to very large classes of functions, both algebraic and transcendental. Leibniz also devised a good notation, much of which is still being used.

Today calculus is essential in engineering and the physical sciences, and is being used more and more in biology and such social sciences as economics, sociology, and psychology. Without calculus, one could not design radar systems or cyclotrons, to name just a few. Calculus is used to determine the orbits of earth satellites and the paths for space travel.

Calculus is generally considered to be one of the greatest intellectual achievements of mankind.

1.2 SETS

A **set** is a collection of things. Some examples of sets are the letters in our alphabet, all American citizens, the positive integers, and the positions on a baseball team.

The **elements** of a set are the objects belonging to the set; they may or may not be material. In this book we shall be chiefly concerned with sets of real numbers and sets of points. The statement **a is an element of the set S** is symbolized by

$$a \in S,$$

and **a is not an element of S** is symbolized by $a \notin S$.

A set is **defined** when its description is sufficient to enable us to determine whether any arbitrary object belongs to the set. For instance, if S is the set of all integers greater than $\frac{4}{3}$, then $7 \in S$, $\frac{9}{4} \notin S$, and $-3 \notin S$. It is essential that if a is any object whatever, the definition of a set will enable us to give the unqualified answer “Yes” or “No” to the question “Does a belong to the set?” Thus “all beautiful women” fails to define a set because the decision of membership would be a matter of opinion.

When the number of elements of a set is finite, we can define the set by listing its elements. For example, the set consisting of the numbers π , $\sqrt{2}$, and 8 can be written $\{\pi, \sqrt{2}, 8\}$. Other sets are $\{a, b, c, d\}$ and $\{2, 3, 5, 7, 11\}$.

A different kind of set is

$$S = \{-1, 6\}, \{8, 16, 24\}, \{z, w\}.$$

This is a **set of sets** (or a collection of sets) whose three elements are the sets $\{-1, 6\}$, $\{8, 16, 24\}$, and $\{z, w\}$. Notice that $-1 \notin S$, although $\{-1, 6\} \in S$.

If the number of elements of a set is not finite, or if it is not convenient to list all the elements of a set, some rule that enables us to determine whether any given object belongs to the set will suffice. To illustrate, “the set of all numbers that can be expressed in the form $2n$, where n is an integer” defines the set of even integers.

The symbol

$$\{x | \dots\}$$

means “the set of elements x such that”; the three dots here stand for some statement or statements about the elements of the set that clearly define the set. For example, $\{x | x \text{ is a real number and } 2x^2 - 5x - 3 = 0\}$ is the set of real numbers x such that $2x^2 - 5x - 3 = 0$ is true; in other words, it is the set consisting of the real roots of $2x^2 - 5x - 3 = 0$, which is the set $\{3, -\frac{1}{2}\}$. Again, $\{x | x \text{ is a positive integer and } x \text{ is less than } 10\}$ is the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Another example is $\{x | x \text{ is a negative integer and } x^2 - x - 6 = 0\}$, which is the set $\{-2\}$.

Notice that $\{x | x \text{ is a real number and } x^2 + 1 = 0\}$ contains no elements at all. It is the **empty set** and is represented by \emptyset . Thus $\emptyset = \{x | x \text{ is a real number, } x^2 + 1 = 0\}$, $\emptyset = \{\theta | \sin \theta = 1.5\}$, and $\emptyset = \{y | y \text{ is a living person and } y \text{ signed the Declaration of Independence}\}$.