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Harmonic Measure

Geometric and Analytic Points of View

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Harmonic Measure

Geometric and Analytic Points of View

S'io avessi le rime e aspre e chiocce,
Come si converrebbe al tristo buco,
Sopra 'l quale pontan tutte l'altre rocce,
Io premerei di mio concetto il suco,
Piu pienamente, ma perch'io non l'abbo,
Non senza tema a dicer mi conduco,
Che' non e' impresa da pigliare a gabbo,
Descriver fondo a tutto l'universo,...

Dante, Inferno, Canto 32.

Introduction

This book is based on a series of five lectures that Carlos Kenig gave during the 25th Arkansas Spring Lectures Series in March 2000, at the University of Arkansas.

In these lectures, Kenig described his joint work with Tatiana Toro concerning end-point analogues of the well-known potential theoretic result of Kellogg, which says that the density k of the harmonic measure of a $C^{1,\alpha}$ domain, has logarithm in C^{α}; and of the '*converse*' of this result, the free boundary regularity theorem of Alt-Caffarelli [**2**], which says that under (necessary) mild hypothesis, if $\log k$ is C^{α}, then the domain must be of class $C^{1,\alpha}$. The potential theoretic results are extensions of the classical function theoretic work of Lavrentiev [**53**] and Pommerenke [**61**], and the higher dimensional results of Dahlberg [**16**] and Jerison-Kenig [**34**].

The free boundary results, on the one hand, give a geometric measure theoretic characterization of the support sets of measures which are "*asymptotically optimally doubling*" in terms of "*flatness*" conditions on the support, and extend the Alt-Caffarelli higher dimensional version [**2**] of the "*converse*" result of Pommerenke's [**61**], to the end-point VMO case. This type of end-point version of the Alt-Caffarelli result was first introduced by David Jerison [**32**].

The book follows closely the format of the lectures. In particular, for each of the main Theorems in Chapter 6 and in the first section of Chapter 7, we present a short "sketch of the proof" which is an almost verbatim copy of the argument described in the lectures. These brief sketches are followed by detailed proofs. In this way we hope to communicate the main ideas and convey the enthusiasm and the intuitive insight which made the lectures so lively and exciting.

We break this pattern in the proof of the last two theorems (Sections two and three in Chapter 7), for which the sketch of the proof alone is already quite long and technically involved. The interested reader will find details for these theorems in [**45**] and [**46**]. We hope that our presentation will provide a "reading key" to help navigate through these papers.

In order to make the presentation more self-contained and comprehensive, a review of the classical results for planar domains has been added in Chapter 2, where conformal mapping is the main tool to approach the problems.

Kenig would like to thank T. Toro for her fundamental contribution to their joint work and D. Jerison for many conversations on the subject throughout the years. Kenig would also like to thank Luis Caffarelli and Guy David for useful discussions, and G. David for his role in their joint work in the higher co-dimension case of the geometric measure theory results.

We are indebted to Joan Carmona, Christian Pommerenke, and Joan Verdera for discussing with us many of the two-dimensional results. It is a pleasure to thank Chaim Goodman-Strauss for producing the pictures in the book, and Christine Thiverge at the American Mathematical Society for her assistance with this project.

Last but not least, the authors wish to thank the National Science Foundation and the University of Arkansas for sponsoring the 2000 Arkansas Spring Lectures Series.

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Contents

CHAPTER 1

Motivation and statement of the main results

This chapter can be read separately from the rest of the book; it is intended to give a quick overview of the results presented in the next chapters, and to place them within the current state of knowledge in potential theory and boundary value problems.

As motivation for the the topics to be discussed in these notes, let us consider the solution to the classical *Dirichlet Problem* for a connected, open set $\Omega \subset \mathbb{R}^n$, i.e. the unique, smooth function $u \in C^\infty(\Omega)$, satisfying

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \in C_b(\partial\Omega). \end{cases} \tag{1.1}$$

Here $C_b(\partial\Omega)$ denotes the space of continuous, bounded functions defined on $\partial\Omega$. The boundary $\partial\Omega$ is called *regular* if in addition $u \in C_b(\overline{\Omega})$ and achieves the boundary data continuously. The maximum principle yields, via the Riesz representation theorem, the representation formula

$$u(x_*) = \int_{\partial\Omega} f(Q)d\omega^{x_*}(Q), \text{ for every } x_* \in \Omega, \tag{1.2}$$

where the family of probability, positive measures $\{d\omega^{x_*}\}$ is the *harmonic measure* corresponding to the Dirichlet problem in Ω. For the precise definition of $\{d\omega^{x_*}\}$ see [**42**, Definition 1.2.6]. When there is no risk of ambiguity, we fix $x_* \in \Omega$ and denote $d\omega = d\omega^{x_*}$. Roughly speaking, the smoothness of the domain determines the smoothness of solutions to the Dirichlet problem. If the domain Ω is "sufficiently regular" (see for instance Theorem 1.14, Theorem 6.11, [**16**] or [**18**]) then harmonic measure and surface measure $d\sigma$ are mutually absolutely continuous. In this case we denote the *Poisson kernel* for the domain Ω by

$$k(\cdot, x_*) = \frac{d\omega^{x_*}}{d\sigma}.$$

For instance, if Ω is a smooth domain then

$$d\omega^{x_*}(Q) = \frac{\partial G}{\partial \vec{n}_Q}(Q, x_*)d\sigma(Q), \tag{1.3}$$

where G denotes the Green function for Ω, $\vec{n}_Q$ is the outer normal at $Q \in \partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$. For more details see e.g. [**30**, Sec. 2.4]. Let Ω be an unbounded domain. If a function $v \in C^\infty(\Omega)$ satisfies

$$(1.4) \qquad \begin{cases} \Delta v = 0 \text{ in } \Omega \\ v > 0 \text{ in } \Omega, \\ v|_{\partial\Omega} = 0, \end{cases}$$

then we call it *minimal positive harmonic function* in Ω. We denote by $d\omega^\infty$ the *harmonic measure with pole at infinity,*

$$d\omega^\infty = \frac{\partial v}{\partial \vec{n}_Q}(Q)d\sigma(Q).$$

Here the function v plays the same role as the Poisson kernel for a bounded domain. As an example, if $\Omega = \mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} \mid t > 0\}$, then $v(x,t) = t$ and $d\omega^\infty = dx$ on $\mathbb{R}^n$. As we show below, the classes of domains that are studied in this book are so general that the classical surface measure may not be well defined. In that case we substitute $d\sigma$ with the restriction to $\partial\Omega$ of the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

Two basic (and related) questions we intend to address are:

Question 1. *What is the relationship between the regularity of the domain and the doubling character of its harmonic measure?*
Question 2. *What is the relationship between the regularity of the domain and the smoothness of its Poisson kernel?*

In particular, the results we are going to describe originated from trying to understand in the asymptotic limit as $\alpha \to 0$, the following well-known results:

THEOREM 1.1 (Kellogg [**40**]). *If $\Omega \subset \mathbb{R}^n$ is of class $C^{1,\alpha}$, $0 < \alpha < 1$, then $d\omega = kd\sigma$, and $\log k \in C^\alpha$,*

and its converse,

THEOREM 1.2 (Alt and Caffarelli [**2**]). *If $\Omega \subset \mathbb{R}^n$ satisfies certain (necessary) "weak conditions" (to be specified later) and $\log k \in C^\alpha$, then Ω is of class $C^{1,\alpha}$.*

1. Characterization $(1)_\alpha$: Approximation with planes

The first step in answering Questions 1 and 2 consists in finding the correct formulation of the results above, as $\alpha \to 0$. To do this, we need to recall two real-variable characterizations of the Hölder classes C^α and $C^{1,\alpha}$:

Let $0 < \alpha < 1$ and ϕ be real-valued. We say that $\phi \in C^{1,\alpha}(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for any $r > 0$ and $x_0 \in \mathbb{R}^n$ there is an affine function L_{r,x_0} on $\mathbb{R}^n$ satisfying

$$(1.5) \qquad \frac{|\phi(x) - L_{r,x_0}(x)|}{r} \leq Cr^\alpha, \qquad \text{for } |x - x_0| < r.$$

At $\alpha = 0$, we have $(1)_0$, which has the equivalent formulation (*Zygmund's Λ_* class*)

$$(1.6) \qquad \frac{|\phi(x+h) + \phi(x-h) - 2\phi(x)|}{|h|} \leq C, \qquad \text{for every } x \in \Omega.$$

The *Weierstrass nowhere differentiable function*

$$\sum_{k=0}^{\infty} \frac{\sin(\pi 2^k x)}{2^k}$$

is a typical element of the Zygmund class Λ_*. Note that if ϕ is differentiable at the point x, then the left hand side of (1.6) vanishes as $|h| \to 0$. This leads to the so-called *Zygmund* λ_* *class.*

DEFINITION 1.3. *A real valued function* $\phi \in \Lambda_*$ *is in* λ_* *if*

$$\lim_{|h|\to 0} \frac{|\phi(x+h) + \phi(x-h) - 2\phi(x)|}{|h|} = 0 \tag{1.7}$$

uniformly in x.

Functions in this class may be quite irregular, for instance

$$\phi(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}}$$

is in λ_* and is almost everywhere not differentiable (see [**78**, pg. 47]).

2. Characterization $(2)_\alpha$: Introducing BMO and VMO

We want to characterize $C^{1,\alpha}$ domains. To this end we say that Ω is $C^{1,\alpha}$ if and only if its outward unit normal $\vec{n}$ is in C^α. It is known that a vector valued function $h : \mathbb{R} \to \mathbb{R}^n$ is in $C^\alpha(\mathbb{R})$ if and only if

$$\sup_{r>0,\ |I|=r} \frac{1}{r^\alpha} \frac{1}{|I|} \int_I |h - h_I| dx \leq C, \tag{1.8}$$

where I denote intervals, $|I|$ their length, and h_I denotes the average of h on I (see [**29**] and references therein). If we let $\alpha = 0$ in the above condition, we obtain the space of functions of bounded mean oscillation (BMO) of John and Nirenberg [**38**]. However such definition may not be used to measure the regularity of the unit normal since $|\vec{n}| = 1$. Hence we introduce the so-called VMO (*Vanishing Mean Oscillation*) class, where $h \in VMO$ if and only if $h \in BMO$ and

$$\lim_{r\to 0,\ |I|=r} \frac{1}{|I|} \int_I |h - h_I| dx = 0.$$

The class VMO plays the same role vis a vis BMO that continuous functions play with respect to the L^∞ space.

3. Multiplicative vs. additive formulation: Introducing the doubling condition

Conditions $(1)_\alpha$, and $(2)_\alpha$ are in a certain sense "*additive*". We will also need corresponding "*multiplicative conditions*" (fitted to Borel measures), see for instance [**42**, Pg. 77]. For example: If $\phi : \mathbb{R} \to \mathbb{R}$, and ϕ' exists, then (1.6) and (1.3) can be formulated in the following way

$$\phi \in \Lambda_* \quad \text{if and only if} \quad \left| \frac{1}{|I_r|}\int_{I_r} \phi' - \frac{1}{|I_l|}\int_{I_l} \phi' \right| \leq C, \tag{1.9}$$

$$\phi \in \lambda_* \quad \text{if and only if} \quad \left| \frac{1}{|I_r|}\int_{I_r} \phi' - \frac{1}{|I_l|}\int_{I_l} \phi' \right| \leq o(1), \tag{1.10}$$

where $I_r = [x, x+h]$, and $I_l = [x-h, x]$. In order to state the corresponding "*multiplicative conditions*" we define positive Borel measures $d\omega = e^{\phi'}dx$ and assume for a moment that, with respect to the weight ϕ', the operations of exponentiation and averaging can be commuted, that is $⨍_I \exp\phi' \approx \exp ⨍_I \phi'$ (here $⨍_I$ denotes average over I and $\approx$ denotes a two-sided bound with multiplicative constants). This is obviously not true in general, but for instance it holds if $\exp\phi' \in A_\infty(dx)$ (see **[25]** for more details). Given this assumption, then (1.9), and (1.10) can be rephrased as follows:

$$(\Lambda_* \text{ class}) \quad \text{For any } x \text{ in } \mathbb{R}, \text{ one has } \frac{\omega(I_r)}{\omega(I_l)} \leq C, \tag{1.11}$$

$$(\lambda_* \text{ class}) \quad \text{For any } x \text{ in } \mathbb{R}, \text{ one has } \frac{\omega(I_r)}{\omega(I_l)} \leq 1 + o(1), \text{ as } |h| \to 0. \tag{1.12}$$

Conditions (1.11) and (1.12) give a rough notion of "regularity" for the measure $d\omega$. In fact, (1.11) is equivalent to $d\omega$ being a *doubling* measure (see below for the precise definition), while (1.12) is in some sense an *optimal doubling* condition. However, the reader should keep in mind that in general such measures could be purely singular with respect to the Lebesgue measure dx on $\mathbb{R}$, see **[5]** and **[9]**. The "*multiplicative*" analogue, in terms of ω, of condition $(2)_0$ is given by $d\omega = kdx$ and $\log k \in VMO$ (see M. Korey **[49]** and **[50]**).

4. Characterization $(1)_\alpha$ and flatness

Next, we introduce a geometric version of $(1)_0$, namely the notion of *"Locally flat domains"*. This will allow us to state some geometric measure theory results which have $(1)_0$ as a point of departure.

We begin by recalling the definition of *Hausdorff distance* D between two subsets A, B of $\mathbb{R}^{n+1}$ (see also **[24]**): We say that $D[A,B] \leq \delta$ if and only if A is in a $\delta-$neighborhood of B and B is in a $\delta-$neighborhood of A, i.e.

$$D[A,B] = \max\Big(\sup\{d(a,B) | a \in A\}; \sup\{d(A,b) | b \in B\} \Big). \tag{1.13}$$

DEFINITION 1.4. *We say that $\Omega \subset \mathbb{R}^{n+1}$ is $\delta-$Reifenberg flat if and only if for every compact set $K \subset \mathbb{R}^{n+1}$, there exists $R_K > 0$ such that if $Q \in K \cap \partial\Omega$ and $0 < r < R_K$, then there exists an $n-$dimensional plane $L(r,Q)$ containing Q and such that*

$$\frac{1}{r} D[\ B(r,Q) \cap \partial\Omega\ ;\ B(r,Q) \cap L(r,Q)\ ] \leq \delta. \tag{1.14}$$

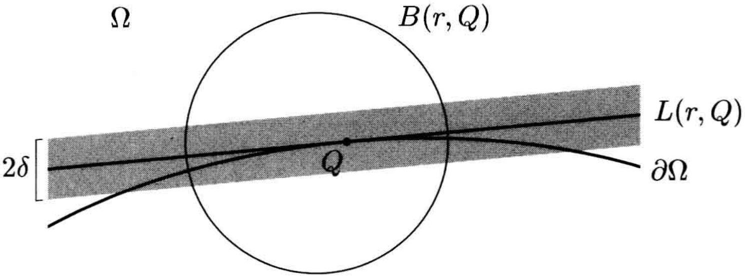

FIGURE 1. Reifenberg flat domain.

Note that this definition is significant only for δ small. We will always assume $\delta < \frac{1}{8}$ and motivate this choice later. Set

$$\theta(r,Q) = \inf_{Q \in L(r,Q)} \left\{ \frac{1}{r} D[\ B(r,Q) \cap \partial\Omega\ ;\ B(r,Q) \cap L(r,Q)\ ] \right\} \tag{1.15}$$

and

$$\theta_K(r) = \sup_{Q \in \partial\Omega \cap K} \theta(r,Q). \tag{1.16}$$

We say that Ω is *vanishing Reifenberg*, if it is $\delta-$Reifenberg flat ($\delta < 1/8$), and moreover, for every compact set $K \subset \mathbb{R}^{n+1}$ we have

$$\limsup_{r \to 0} \theta_K(r) = 0. \tag{1.17}$$

In a certain sense, such domains are *"locally flat"*. To simplify the notation, when there is no ambiguity we will denote $L(r,Q)$ by L_Q.

To help clarify the definitions of Reifenberg flat and vanishing Reifenberg domains we present a few examples.

We begin with the two most basic examples, namely the disc and the wedge, that will emphasize the relevance of the *vanishing condition* (1.17). Such examples serve as the prototype of the classes of smooth and Lipschitz domains respectively (see Definition 3.15).

Example 4.1. For the wedge centered at a point w and with angle $0 < \psi < \pi$, we let $\beta = (\pi - \psi)/2$, see figure 2. Elementary computations yield $\inf_{L(r,w)} \theta(r,w) = r \sin\beta$. This shows that as the wedge opens up, converging to a line, the domain becomes vanishing Reifenberg. On the other hand, the ratio $\theta(r,w)/r$ cannot be made smaller by decreasing the scale r.

Example 4.2. In the case of the disc of radius $R > 0$ we let w denote a boundary point, see figure 3. Choosing $L(r,w)$ to be the tangent we derive the formula

$$\theta(r,w) = \max\left\{ \frac{r^2}{2R^2}; \sqrt{R^2 + r^2} - R \right\},$$

from which it immediately follows that the disc is vanishing Reifenberg.

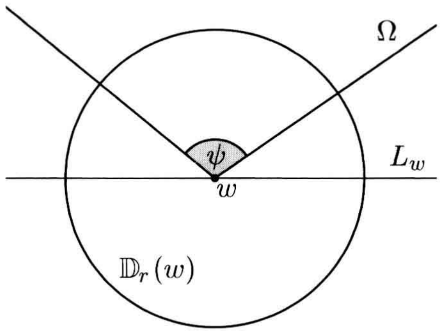

FIGURE 2. The wedge: $\inf_{L(r,w)} \theta(r,w) = r \sin\beta$.

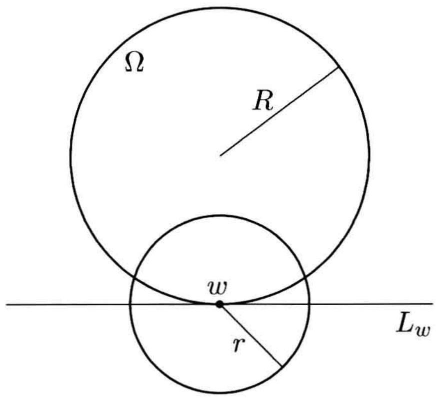

FIGURE 3. The disc: $\theta(r,w) = \max\left\{\frac{r^2}{2R^2}; \sqrt{R^2+r^2} - R\right\}$.

Example 4.3. Every domain Ω above the graph of a function $\phi \in \lambda_*$ is vanishing Reifenberg (see the proof of Corollary 6.16). This, in particular, shows that vanishing Reifenberg domains need not be Lipschitz regular. Recall that $\phi(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}}$, is in the λ_* Zygmund class (see [**78**, pg. 47]), but it is almost nowhere differentiable. Hence, vanishing Reifenberg flat domains may be even less regular than Lipschitz, and may not have a classical "surface measure".

Example 4.4. In general, vanishing Reifenberg domains do not have tangent planes anywhere on their boundary (as we have already seen), and may not even be local graphs. For instance, in dimension $n = 1$ we may consider the snowflake curve (see [**20**]), with angles tending to zero sufficiently slowly. This provides an example of a Reifenberg flat set with vanishing constant which has locally infinite $\mathcal{H}^1$ Hausdorff measure. More precisely, let us recall an argument from [**75**]:

Define the set S_β to be the self-similar snowflake which is obtained by applying iteratively a set of contractions and rotations to a generating curve which, in our case, is the wedge in Example 4.1, with angle $0 < \psi < \pi$. Set $\beta = (\pi - \psi)/2$. This is the same iterative process that leads to the classical Koch curve $S_{\pi/3}$. In