ADVANCED ENGINEERING MATHEMATICS

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FOURTH EDITION

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PREFACE

The first edition of this book was written to provide an introduction to those branches of postcalculus mathematics with which the average analytical engineer or physicist needs to be reasonably familiar in order to carry on his own work effectively and keep abreast of current developments in his field. In the present edition, as in the second and third, although the material has been largely rewritten, the various additions, deletions, and refinements have been made only because they seemed to contribute to the achievement of this goal.

Because ordinary differential equations are probably the most immediately useful part of postcalculus mathematics for the student of applied science, and because the techniques for solving simple ordinary differential equations stem naturally from the techniques of calculus, this book begins with a chapter on ordinary differential equations of the first order and their applications. This is followed by two other chapters on ordinary differential equations, which develop the theory and applications of linear equations and systems of linear equations with constant coefficients. In particular, a new section on Green's function and its interpretation as an influence function has been added to the first of these two chapters. Following these is a chapter on finite differences containing the usual applications to interpolation, numerical differentiation and integration, and the step-by-step solution of differential equations using both Milne's method and the Runge-Kutta method. There is also a section on linear difference equations with constant coefficients closely paralleling the preceding development for differential equations and a section on the method of least squares and the related topic of orthogonal polynomials. It is hoped that the material in this chapter will provide a useful background in classical finite differences, on which a more extensive course in computer-oriented numerical analysis can be based. Chapter 5 is devoted to the application of the preceding ideas to mechanical and electrical systems, and, as in the first three editions, the mathematical identity of these fields is emphasized. The next two chapters deal first with partial differential equations and boundary-value problems and second with Bessel functions and Legendre polynomials, very much as in the third edition though with a number of new examples and exercises.

Chapters 10 and 11 deal with determinants and matrices as far as the Cayley-Hamilton theorem, Sylvester's identity, and infinite series of matrices and their use in solving matric differential equations. Chapter 12 is a new chapter on the calculus of variations, covering such topics as the maxima and minima of functions of several variables, Lagrange's multipliers, the extremal properties of the eigenvalues of matric equations, Euler's equation, Hamilton's principle, and Lagrange's equation. Chapters

13 and 14 deal with vector and tensor analysis. The last four chapters provide an introduction to the theory of functions of a complex variable, with applications to the evaluation of real definite integrals, the complex inversion integral, stability criteria, conformal mapping, and the Schwarz-Christoffel transformation.

This book falls naturally into three major subdivisions. The first nine chapters constitute a reasonably self-contained treatment of ordinary and partial differential equations and their applications. The next five chapters cover the related areas of linear algebra, the calculus of variations, and vector and tensor analysis. The last four chapters cover the elementary theory and applications of functions of a complex variable. With this organization, the book, which contains enough material for a 2-year postcalculus course in applied mathematics, is well adapted for use as a text for any of several shorter courses.

In this edition, as in the first three, every effort has been made to keep the presentation detailed and clear while at the same time maintaining acceptable standards of precision and accuracy. To achieve this, more than the usual number of worked examples and carefully drawn figures have been included, and in every development there has been a conscious attempt to make the transitions from step to step so clear that a student with no more than a good background in calculus, working with paper and pencil, should seldom be held up more than momentarily. Over 750 new exercises of varying degrees of difficulty have been added to the 1,387 problems which appeared in the third edition. Many of these involve extensions of topics presented in the text or related topics which could not be treated because of limitations of space. Hints are included in many of the exercises, and answers to the odd-numbered ones are given at the end of the book. As in the first three editions, words and phrases defined informally in the body of the text are set in boldface. Illustrative examples are set in type of a different size from that used for the main body of the text.

The indebtedness of the author to his colleagues, students, and former teachers is too great to catalog, and to all who have given help and encouragement in the preparation of this book, I can offer here only a most inadequate acknowledgment of my appreciation. In particular, I am deeply grateful to those users of this book, both teachers and students, who have been kind enough to write me their impressions and criticisms of the first three editions and their suggestions for an improved fourth edition. Finally, I must express my gratitude to my wife Ellen and my student Moffie Hills, who have shared with me the task of proofreading the manuscript in all its stages.

C. Ray Wylie

TO THE STUDENT

This book has been written to help you in your development as an applied scientist, whether engineer, physicist, chemist, or mathematician. It contains material which you will find of great use, not only in the technical courses you have yet to take, but also in your profession after graduation as long as you deal with the analytical aspects of your field.

I have tried to write a book which you will find not only useful but also easy to study from, at least as easy as a book on advanced mathematics can be. There is a good deal of theory in it, for it is the theoretical portion of a subject which is the basis for the nonroutine applications of tomorrow. But nowhere will you find theory for its own sake, interesting and legitimate as this may be to a pure mathematician. Our theoretical discussions are designed to illuminate principles, to indicate generalizations, to establish limits within which a given technique may or may not safely be used, or to point out pitfalls into which one might otherwise stumble. On the other hand, there are many applications illustrating, with the material at hand, the usual steps in the solution of a physical problem: formulation, manipulation, and interpretation. These examples are, without exception, carefully set up and completely worked, with all but the simplest steps included. Study them carefully, with paper and pencil at hand, for they are an integral part of the text. If you do this, you should find the exercises, though challenging, still within your ability to work.

Terms defined informally in the body of the text are always indicated by the use of boldface type. Italic type is used for emphasis. It is suggested that you read each section through for the main ideas before you concentrate on filling in any of the details. You will probably be surprised at how many times a detail which seems to hold you up in one paragraph is explained in the next as the discussion unfolds.

Because this book is long and contains material suitable for various courses, your instructor may begin with any of a number of chapters. However, the overall structure of the book is the following: The first nine chapters are devoted to the general theme of ordinary and partial differential equations and related topics. Here you will find basic analytical techniques for solving the equations in which physical problems must be formulated when continuously changing quantities are involved. Chapters 10 through 14 deal with the somewhat related topics of matrix theory and linear algebra, the calculus of variations, vector analysis, and an introduction to generalized coordinates and tensor analysis. Finally, Chapters 15 to 18 provide an introduction to the theory and applications of functions of a complex variable. (Chapter 4, in particular, is worthy of note because it provides an introduction to numerical analysis, the modern field which deals with techniques for obtaining numerical answers to problems too complicated to be solved by exact analytic methods.)

It has been gratifying to receive letters from students who have used this book, giving me their reactions to it, pointing out errors and misprints in it, and offering suggestions for its improvement. Should you be inclined to do so, I should be happy to hear from you also. And now good luck and every success.

C. Ray Wylie

CONTENTS

CHAPTER 1
Ordinary Differential Equations of the First Order

1.1 Introduction

An equation involving one or more derivatives of a function is called a **differential equation.** By a **solution** of a differential equation is meant a relation between the dependent and independent variables which is free of derivatives and which, when substituted into the given equation, reduces it to an identity. The study of the existence, nature, and determination of solutions of differential equations is of fundamental importance not only to the pure mathematician but also to anyone engaged in the mathematical analysis of natural phenomena.

In general, a mathematician considers it a triumph if he is able to prove that a given differential equation possesses a solution and if he can deduce a few of the more important properties of that solution. A physicist or engineer, on the other hand, is usually greatly disappointed if a specific expression for the solution cannot be exhibited. The usual compromise is to find some practical procedure by means of which the required solution can be approximated with satisfactory accuracy.

Not all differential equations are difficult enough to make this necessary, however, and there are several large and very important classes of equations for which solutions can readily be found. For instance, an equation such as

$$\frac{dy}{dx} = f(x)$$

is really a differential equation, and the integral

$$y = \int f(x)\,dx + c$$

is a solution. More generally, the equation

$$\frac{d^n y}{dx^n} = g(x)$$

is a differential equation whose solution can be found by n successive integrations. Except in name, the process of integration is actually an example of a process for solving differential equations.

In this and the following two chapters we shall consider differential equations which are next in difficulty after those which can be solved by direct integration. These equations form only a very small part of the class of all differential equations, and yet with a knowledge of them a scientist is equipped to handle a great variety of applications. To get so much for so little is indeed remarkable.

1.2 Fundamental Definitions

If the derivatives which appear in a differential equation are total derivatives, the equation is called an **ordinary differential equation**; if partial derivatives occur, the equation is called a **partial differential equation.** By the **order** of a differential equation is meant the order of the highest derivative which appears in the equation.

EXAMPLE 1

The equation $x^2y'' + xy' + (x^2 - 4)y = 0$ is an *ordinary* differential equation of the *second* order connecting the dependent variable y with its first and second derivatives and with the independent variable x.

EXAMPLE 2

The equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial x^2\, \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

is a *partial* differential equation of the *fourth* order.

At present we shall be concerned exclusively with ordinary differential equations.

An equation which is linear, i.e., of the first degree, in the *dependent* variable and its derivatives is called a **linear differential equation.** From this definition it follows that the most general (ordinary) linear differential equation of order n is of the form

$$(1) \qquad p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = r(x)$$

A differential equation which is not linear, i.e., cannot be put in the form (1), is said to be **nonlinear**. In general, linear equations are much easier to solve than nonlinear ones, and most elementary applications involve linear equations.

EXAMPLE 3

The equation $y'' + 4xy' + 2y = \cos x$ is a *linear* equation of the second order. The presence of the terms xy' and $\cos x$ does not alter the fact that the equation is linear, because, by definition, linearity is determined solely by the way the *dependent* variable y and its derivatives enter into combination among themselves.

EXAMPLE 4

The equation $y'' + 4yy' + 2y = \cos x$ is a *nonlinear* equation because of the occurrence of the product of y and one of its derivatives.

EXAMPLE 5

The equation $y'' + \sin y = 0$ is *nonlinear* because of the presence of $\sin y$, which is a nonlinear function of y.

As illustrated by the simple differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

and its solution

$$y = \int e^{-x^2}\, dx + c$$

the solution of a differential equation may depend upon integrals which cannot be evaluated in terms of elementary functions. This example also illustrates the fact that a solution of a differential equation usually involves one or more arbitrary constants.

A detailed treatment of the question of the maximum number of *essential* arbitrary constants that a solution of a differential equation may contain or even of what is meant by essential constants is quite difficult.† For our purposes, if an expression contains n arbitrary constants, we shall consider them essential if they cannot, through formal rearrangement of the expression, be replaced by any smaller number of constants. For example,

$$a \cos^2 x + b \sin^2 x + c \cos 2x \tag{2}$$

contains three arbitrary constants. However, since

$$\cos 2x = \cos^2 x - \sin^2 x$$

the expression (2) can be written in the form

$$\begin{aligned} a \cos^2 x + b \sin^2 x + c(\cos^2 x - \sin^2 x) &= (a + c) \cos^2 x + (b - c) \sin^2 x \\ &= d \cos^2 x + e \sin^2 x \end{aligned}$$

where $d = a + c$ and $e = b - c$. The fact that the three arbitrary constants a, b, and c can be replaced by the two constants d and e shows that the former are not all essential. On the other hand, since $\cos^2 x$ and $\sin^2 x$ are linearly independent‡ (whereas $\cos^2 x$, $\sin^2 x$, and $\cos 2x$ are linearly dependent), it follows that there is no further rearrangement of the given expression that will permit d and e to be combined into, and replaced by, a single new arbitrary constant. Hence d and e are essential.

It is frequently the case (especially with linear equations) that a differential equation of order n possesses solutions containing n essential arbitrary constants, or parameters, but none containing more. However, there are equations such as

$$\left|\frac{dy}{dx}\right| + |y| = 0$$

(which clearly has only the single solution $y = 0$) and

$$\left|\frac{dy}{dx}\right| + 1 = 0$$

(which has no solutions at all) which possess *no* solutions containing *any* arbitrary constants. Moreover, there are also simple differential equations which possess solutions containing more essential parameters than the order of the equation. For instance it is easy to verify that the arc of the family $y = c_1 x^2$ $(x \leq 0)$ (Fig. 1.1*a*) corresponding to any value of c_1 can be paired with the arc of the family $y = c_2 x^2$ $(x \geq 0)$ (Fig. 1.1*b*) corresponding to any value of c_2, to give a function which satisfies the differential equation

$$xy' = 2y \tag{3}$$

† See, for instance, R. P. Agnew, "Differential Equations," 2d ed., pp. 103–105, McGraw-Hill, New York, 1960.

‡ See Definitions 1 and 2, Sec. 10.5.

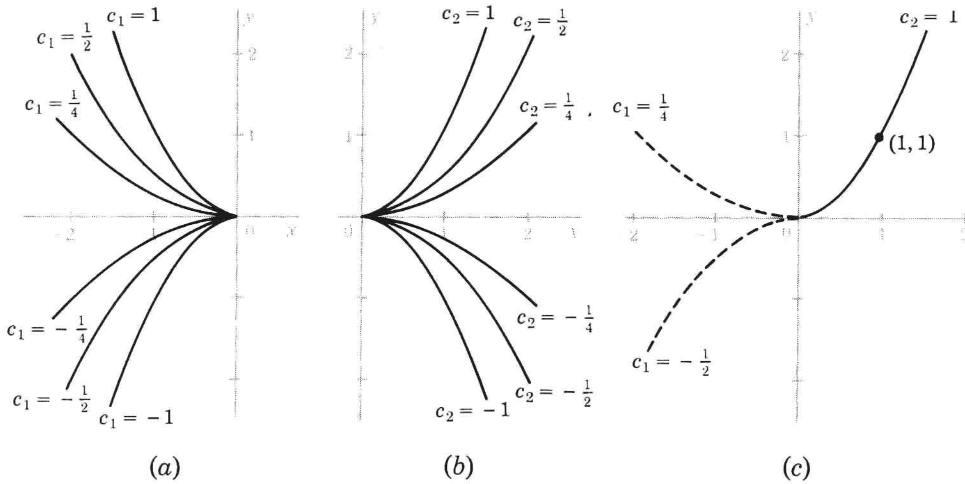

Figure 1.1
Arcs of different parabolas of the family $y = cx^2$ pieced together to give solutions of the differential equation $xy' = 2y$.

for all values of x (Fig. 1.1*c*). In other words, for all choices of the *two* essential constants c_1 and c_2, the rule

$$y = \begin{cases} c_1x^2 & x \leq 0 \\ c_2x^2 & x > 0 \end{cases}$$

defines a function which is continuous and differentiable for all values of x and which satisfies the equation (3) over the entire x axis. A still more striking example of this sort appears in Exercise 44, where a first-order equation with a solution containing infinitely many essential parameters is given.

As the foregoing suggests, it is difficult, if not impossible, to make statements valid for all differential equations. The theory of differential equations is essentially a body of theorems concerning particular classes of equations defined by such considerations as the order and linearity of the equation and the continuity and boundedness of its coefficients. Typical of these theorems is the following result,† which is of fundamental importance in the theory of the equations we shall consider in this chapter, namely, equations of the first order.

THEOREM 1 Let (x_0,y_0) be a point of the xy plane; let R be the rectangular region defined by the inequalities $|x - x_0| \leq a$, $|y - y_0| \leq b$; let $f(x,y)$ and $f_y(x,y) \equiv \partial f(x,y)/\partial y$ be single-valued and continuous at all points of R; let M be a constant such that $|f(x,y)| < M$ at all points of R; and let h be the smaller of the numbers a and b/M. Then, on the interval $|x - x_0| < h$, there is a unique continuous function y which satisfies the equation $y' = f(x,y)$ and takes on the value y_0 when $x = x_0$.

† See, for instance, M. Golomb and M. E. Shanks, "Elements of Ordinary Differential Equations," 2d ed., pp. 63–78, McGraw-Hill, New York, 1965.

It is instructive to reconsider Eq. (3) in the light of Theorem 1. For this equation we have $f(x,y) = 2y/x$, and, clearly, neither f nor f_y exists when $x = 0$. Hence, it follows from Theorem 1 that over an interval containing $x = 0$ neither the existence nor the uniqueness of a solution of Eq. (3) can be guaranteed. Actually, as our earlier discussion pointed out, Eq. (3) does have solutions which are valid for all values of x. However, as Fig. 1.1*c* illustrates, over any interval which contains $x = 0$, the solution curve which passes through a given point (x_0, y_0), for example, (1,1), is not unique. On the other hand, according to Theorem 1, over any interval which contains x_0 but does not contain $x = 0$, the solution curve which passes through a given point (x_0, y_0) is unique.

Almost all applications of differential equations involve equations which possess solutions containing at least one arbitrary constant, and for such equations it is convenient to introduce the following definitions. A solution which contains at least one arbitrary constant is called a **general solution**. A solution obtained from a general solution by assigning particular values to the arbitrary constants which appear in it is called a **particular solution**. Solutions which cannot be obtained from any general solution by assigning specific values to its arbitrary constants are called **singular solutions**. If a general solution has the property that *every* solution of the differential equation can be obtained from it by assigning suitable values to its arbitrary constants, it is said to be a **complete solution**. A general solution can thus be thought of as a description of some family of particular solutions, and a complete solution can be thought of as a description of the set of all solutions of the given equation.

It is important to note that we speak of *a* general solution and *a* complete solution of a differential equation and not of *the* general solution and *the* complete solution. If an equation has a general solution or a complete solution, it has many such solutions, and these may differ significantly in form. Moreover, in particular problems involving differential equations, the choice of which complete solution to use often has an important bearing on the ease with which the problem can be solved.

EXAMPLE 6

Verify that $y = ae^{-x} + be^{2x}$ is a solution of the equation $y'' - y' - 2y = 0$ for all values of the constants a and b.

By differentiating y, substituting into the differential equation as indicated, and then collecting terms on a and b, we obtain

$$\begin{aligned}(ae^{-x} + 4be^{2x}) - (-ae^{-x} + 2be^{2x}) - 2(ae^{-x} + be^{2x})\\ &= (e^{-x} + e^{-x} - 2e^{-x})a + (4e^{2x} - 2e^{2x} - 2e^{2x})b\\ &= 0a + 0b = 0\end{aligned}$$

for all values of a and b. Thus, $y = ae^{-x} + be^{2x}$ is a general solution of $y'' - y' - 2y = 0$. In fact, as we shall see in Sec. 2.2, it is a complete solution of this equation.

It is interesting to note that although $y_1 = ae^{-x}$ and $y_2 = be^{2x}$ also satisfy the equation $yy'' - (y')^2 = 0$, the sum $y = y_1 + y_2 = ae^{-x} + be^{2x}$ is *not* a solution of $yy'' - (y')^2 = 0$. In fact, differentiating, substituting, and simplifying, we have

$$(ae^{-x} + be^{2x})(ae^{-x} + 4be^{2x}) - (-ae^{-x} + 2be^{2x})^2 \equiv 9abe^{x}$$

and this cannot vanish identically unless either a or b is zero, i.e., unless the sum y consists of just one or the other of the two individual solutions. Roughly speaking, the reason for this difference in behavior is that the equation $y'' - y' - 2y = 0$ is linear, whereas the equation $yy'' - (y')^2 = 0$ is nonlinear. More precisely, as we shall see in