Excursions in MODERN MATHEMATICS

Peter Tannenbaum / Robert Arnold

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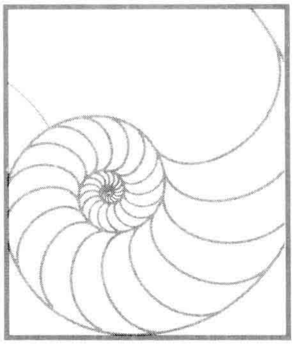

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To my parents, Nicholas and Anna,
and my wife Sally

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To my wife Rachael
and my son Craig

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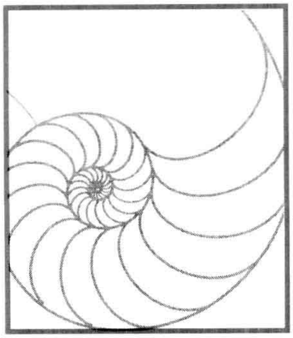

Preface

To most outsiders, modern mathematics is unknown territory. Its borders are protected by dense thickets of technical terms; its landscapes are a mass of indecipherable equations and incomprehensible concepts. Few realize that the world of modern mathematics is rich with vivid images and provocative ideas.

IVARS PETERSON, *The Mathematical Tourist*

Excursions in Modern Mathematics is, as we hope the name might suggest, a collection of "trips" into that vast and alien frontier that many people perceive mathematics to be. While the purpose of this book is quite conventional — it is intended to serve as a textbook for a college-level liberal arts mathematics course — its contents are not. We have made a concerted effort to introduce the reader to a view of mathematics entirely different from the traditional algebra-geometry-trigonometry-finite math curriculum which so many people have learned to dread, fear, and occasionally abhor. The notion that general education mathematics must be dull, unrelated to the real world, highly technical, and deal mostly with concepts that are historically ancient is totally unfounded.

The "excursions" in this book represent a collection of topics chosen to meet a few simple criteria.

- **Applicability.** The connection between the mathematics presented here and down-to-earth, concrete real-life problems is direct and immediate. The often heard question, "What is this stuff good for?" is a legitimate one and deserves to be met head on. The often heard answer, "Well, you need to learn the material in Math 101 so that you can understand Math 102 which you will need to know if you plan to take Math 201 which will teach you the real applications," is less than persuasive and in many cases reinforces students' convictions that mathematics is remote, labyrinthine, and ultimately useless to them.

- **Accessibility.** Sophisticated mathematical topics need not always be highly technical and built on layers of concepts. (Most of us are painfully familiar with the experience of missing or misunderstanding a concept in the middle of a math course and consequently having trouble with the rest of the course material.) In general the choice of topics is such that a heavy mathematical infrastructure is not needed. We have found Intermediate Algebra to be a sufficient prerequisite. In the few instances in which more advanced concepts are unavoidable we have endeavored to provide enough background to make the material self-contained. A word of caution — this does not mean that the material is easy! In mathematics, as in many other walks of life, simple and straightforward is not synonymous with easy and superficial.
- **Age.** Much of the mathematics in this book has been discovered in this century, some as recently as 20 years ago. Modern mathematical discoveries do not have to be only within the grasp of experts.
- **Aesthetics.** The notion that there is such a thing as beauty in mathematics is surprising to most casual observers. There is an important aesthetic component in mathematics and, just as in art and music (which mathematics very much resembles), it often surfaces in the simplest ideas. A fundamental objective of this book is to develop an appreciation for the aesthetic elements of mathematics. It is not necessary that the reader love everything in the book — it is sufficient that he or she find one topic about which they can say, "I really enjoyed learning this stuff!" We believe that anyone coming in with an open mind almost certainly will.

OUTLINE

The material in the book is divided into four independent parts. Each of these parts in turn contains four chapters dealing with interrelated topics.

- **Part I** (Chapters 1 through 4). *The Mathematics of Social Choice.* This part deals with mathematical applications in social science. How do groups make decisions? How are elections decided? How can power be measured? When there are competing interests among members of a group, how are conflicts resolved in a fair and equitable way?
- **Part II** (Chapters 5 through 8). *Management Science.* This part deals with methods for solving problems involving the organization and management of complex activities — that is, activities involving either a large number of steps and/or a large number of variables (building a skyscraper, putting a person on the

moon, organizing a banquet, scheduling classrooms at a big university, etc.). Efficiency is the name of the game in all these problems. Some limited or precious resource (time, money, raw materials) must be managed in such a way that waste is minimized. We deal with problems of this type (consciously or unconsciously) every day of our lives.

- **Part III** (Chapters 9 through 12). *Growth and Symmetry.* This part deals with nontraditional geometric ideas. What do sunflowers and seashells have in common? How do populations grow? What are the symmetries of a pattern? What is the geometry of natural (as opposed to artificial) shapes? What kind of symmetry lies hidden in a cloud?

- **Part IV** (Chapters 13 through 16). *Statistics.* In one way or another, statistics affects all of our lives. Government policy, insurance rates, our health, and our diet are all governed by statistical laws. This part deals with some of the basic elements of statistics. How are statistical data collected? How are they summarized so that they say something intelligible? How are they interpreted? What are the patterns of statistical data?

EXERCISES

We have endeavored to write a book that is flexible enough to appeal to a wide range of readers in a variety of settings. The exercises, in particular, have been designed to convey the depth of the subject matter by addressing a broad spectrum of levels of difficulty — from the routine to the challenging. For convenience we have classified them into three levels of difficulty: **Walking** (these are straightforward applications of the concepts discussed in the chapter, and it is intended that all readers be able to do them); **Jogging** (these are exercises that are not difficult per se, but require a little thinking on the part of the reader); and **Running** (these are the really challenging exercises and are intended for those readers who like a little challenge in life).

TEACHING EXTRAS

Because this course can be taught in a myriad of styles, we have included material which will increase the text's flexibility. Bibliography listings at the end of every chapter can be used by students and instructors as a source of additional readings which provide more indepth or tangentially related information.

In addition, Prentice Hall and the *New York Times* are sponsoring "A Contemporary View," a program which will provide both students and instructors with access to information which further illustrates the relevance and currency of mathematics. Through this program, at the instructor's request, each student will receive a supplement containing recent articles from the *New York Times* which pertain in one way or another to the themes of our excursions in this book. These articles may be used to prompt classroom discussion, to suggest topics for term papers and research projects, or merely to show that the contents of this course have relevance to a well-informed person. Instructors are encouraged to call their Prentice Hall representative for more information.

To ease the instructor's transition to a new book, we have prepared an instructor's manual based on our own experience using the manuscript in class. The instructor's manual contains a test bank consisting of four hundred multiple choice questions as well as answers to all exercises (including worked out solutions to many), helpful notes about teaching emphasis, suggested topic ordering, etc.

A FINAL WORD

This book grew out of the conviction that a textbook of this type should provide much more than just a battery of technical skills. Our ultimate purpose is to instill in the reader something that is more subjective but at the same time more long-lasting — an appreciation of mathematics as a discipline and an exposure to a few of its global paradigms: algorithms as solutions to mathematical problems, the concept of recursion, the idea of complexity and the trade-offs that it entails, the difference between conceptually difficult and procedurally complicated, the idea that mathematics is one of those rare disciplines that can define its own boundaries (impossibility theorems), etc. Last, but not least, we have tried to show that mathematics can be fun.

ACKNOWLEDGMENTS

The writing of this book was a formidable challenge but also a tremendously rewarding experience. We wish to acknowledge, first of all, the many students in our course who shared our excitement for the subject. The many stories we heard of their having rediscovered the joy of doing and learning mathematics confirmed our conviction that this was a worthwhile project.

In the development of this book there were several people whose contributions we specifically wish to acknowledge. The following individuals reviewed various stages of the manuscript and shared many helpful ideas with us:

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Contents

11 Symmetry of Motion. *Mirror, Mirror, off the Wall* 325

12 Symmetry of Scale and Fractals. *Fractally Speaking* 360

PART IV

Statistics

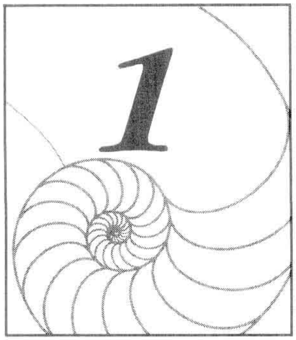
13 Collecting Data. *Censuses, Surveys, and Clinical Studies* 389

14 Descriptive Statistics. *Graphing and Summarizing Data* 419

15 Probability. *Chances Are . . .* 454

16 Normal Distributions. *Everything Is Back to Normal (Almost)* 484

PART I ■ THE MATHEMATICS OF SOCIAL CHOICE



1 Voting

The Paradoxes of Democracy

Democracy is the worst form of government except [for] all those other forms that have been tried from time to time.

WINSTON CHURCHILL

We are often reminded that the right to vote, one of the fundamental tenets of a democracy, is something that we should not take for granted. Voting is the vehicle by which a democratic society makes decisions, and the process by which the many and conflicting opinions of the individuals are consolidated into the single choice of the group is what **voting theory** is about. But why do we need a whole theory? It all sounds so simple. There surely must be a reasonable and easy way for a democratic society to make group decisions based on *numbers*, and we have every right to expect that mathematics with all its tools and power will give us the answer.

Mathematics does indeed provide the answer, but the answer is quite surprising and far from simple: There is *no* consistent method by which a democratic society can make a choice that is *always* fair when that choice must be made from among several (three or more) alternatives. This remarkable fact, discovered by the mathematical economist Kenneth Arrow in 1952, is known as **Arrow's impossibility theorem**. In 1972, Arrow received the Nobel Prize in Economics for his contributions to the mathematical theory of social decision making.

The purpose of this chapter is to clarify the meaning and significance of Arrow's impossibility theorem and at the same time to discuss several well-known methods of voting.

Our discussion for the entire chapter is centered around the following, deceptively simple example.

The MAC Election. An election is being held to choose the president of the Math Anxiety Club (MAC). There are four candidates: Alisha, Boris, Carmen, and Dave (*A*, *B*, *C*, and *D* for short). Each of the 37 members of the club is asked to submit a ballot indicating his or her first, second, third, and fourth choices (ties are not allowed in individual ballots). The 37 ballots submitted are shown in Fig. 1-1. Who should be president? Why?

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *B*	1st *A*	1st *C*	1st *B*	1st *C*	1st *A*	1st *D*	1st *A*	1st *A*
2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *B*
3rd *C*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *C*	3rd *B*	3rd *C*	3rd *C*
4th *D*	4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *D*	4th *D*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *A*	1st *C*	1st *D*	1st *C*	1st *A*	1st *D*	1st *D*	1st *C*	1st *C*
2nd *B*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *B*
3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *D*	3rd *C*	3rd *B*	3rd *B*	3rd *D*	3rd *D*
4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *A*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *B*	1st *C*	1st *C*	1st *D*	1st *A*	1st *D*	1st *C*	1st *A*	1st *D*
2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *C*
3rd *C*	3rd *C*	3rd *D*	3rd *D*	3rd *B*	3rd *C*	3rd *B*	3rd *D*	3rd *C*	3rd *B*
4th *D*	4th *A*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *B*	1st *A*	1st *C*	1st *A*	1st *A*	1st *D*	1st *A*
2nd *D*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*
3rd *C*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*
4th *A*	4th *D*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*

Figure 1-1

PREFERENCE BALLOTS AND PREFERENCE SCHEDULES

Ballots such as the ones shown in Fig. 1-1 in which the voter is asked to list all of the options in order of preference are called **preference ballots**. A closer inspection of the 37 preference ballots in Fig. 1-1 shows that in many instances individual voters voted exactly the same way. In fact, the outcome of the election can be summarized in Fig. 1-2 and more conveniently in the table that follows, which is called a **preference schedule** for the election.

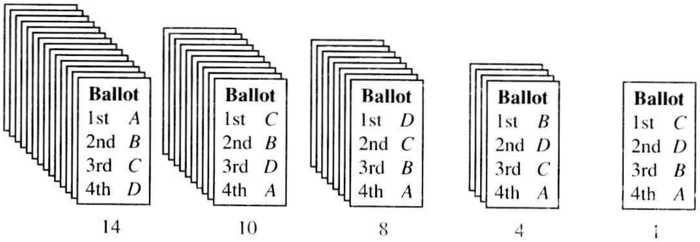


Figure 1-2

Number of voters	14	10	8	4	1
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*
3rd choice	*C*	*D*	*B*	*C*	*B*
4th choice	*D*	*A*	*A*	*A*	*A*

Some Basic Assumptions

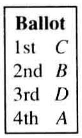


Figure 1-3

Before we take on the task of deciding who will be elected president of the Math Anxiety Club, we need to clarify certain assumptions that we will make about individual preferences. One of our basic assumptions is that individual preferences are **transitive**: If a voter prefers X to Y and Y to Z, then it is only reasonable to assume that the voter prefers X to Z. It follows therefore that a ballot such as the one shown in Fig. 1-3 tells us not only that this voter prefers C to B, B to D, and D to A, but also implicitly that the voter prefers C to D, C to A, and B to A.

A second important assumption is that relative preferences are not altered by the elimination of one or more candidates. In other words, suppose that a voter prefers X to Y, Y to Z, and Z to W. If, for whatever reason, Y drops out of the race, the other relative preferences remain unaffected: The voter still prefers X to Z and Z to W.

While there are occasional situations in which these assumptions do not hold,[1] they are by and large exceptions. In any voting system in which voters act *rationally*, it is reasonable to expect that the above assumptions will hold, and for the rest of this chapter we will pretty much take them for granted.

[1] See, for example, the discussion of how the 1996 Summer Olympics were awarded to Atlanta in Appendix 2 at the end of this chapter.