Dynamics, Ergodic Theory, and Geometry

Boris Hasselblatt
Editor

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Dynamics, Ergodic Theory, and Geometry

Dedicated to Anatole Katok

Edited by

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In this book, which arose from an MSRI research workshop cosponspored by the Clay Mathematical Institute, leading experts give an overview of several areas of dynamical systems that have recently experienced substantial progress.

In symplectic geometry, a fast-growing field having its roots in classical mechanics, Cieliebak, Hofer, Latschev and Schlenk give a definitive survey of quantitative techniques and symplectic capacities, which have become a central research tool. Fisher's survey on local rigidity of group actions is a broad and up-to-date account of a flourishing subject built on the fact that for actions of noncyclic groups, topological conjugacy commonly implies smooth conjugacy.

Other articles by Eigen, Feres, Kochergin, Krieger, Navarro, Pinto, Prasad, Rand and Robinson cover subjects in hyperbolic, parabolic and symbolic dynamics as well as ergodic theory. Among the specific areas of interest are random walks and billiards, diffeomorphisms and flows on surfaces, amenability and tilings.

The articles are complemented by a fifty-page commented problem list, compiled by the editor with the help of numerous specialists. Several sections of this list focus on problems beyond the areas covered in the surveys, and all are sure to inspire and guide further research.

Mathematical Sciences Research Institute
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54

Dynamics, Ergodic Theory, and Geometry

Dedicated to Anatole Katok

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Recent Progress in Dynamics
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Volume **54**, 2007

Foreword

This volume owes its existence to the Clay Mathematics Institute / Mathematical Sciences Research Institute Workshop on "Recent Progress in Dynamics", held at the Mathematical Sciences Research Institute for a week in late September to early October 2004. This is not a proceedings volume, but most authors were participants of the workshop, and the two lead surveys reflect a good deal of what David Fisher and Helmut Hofer presented during their workshop talks.

The workshop represented a broad array of dynamical systems, not least in order to reflect the breadth of taste exhibited by Anatole Katok, whose sixtieth birthday was observed during the workshop. This was possible because we were able to invite a great number of participants from near and far, and the essential ingredient in making this possible was the generous financial support from the Clay Mathematics Institute and the Mathematical Sciences Research Institute (which is in turn supported by the National Science Foundation), as well as from the Pennsylvania State University and Tufts University. As the host of the workshop, the Mathematical Sciences Research Institute also provided administrative support for the organizers and participants. It is a pleasure to acknowledge this support.

Funding alone does not produce a successful workshop, and I want to thank my fellow workshop organizers Michael Brin, Gregory Margulis, Yakov Pesin, Peter Sarnak, Klaus Schmidt, Ralf Spatzier and Robert Zimmer. Foremost among these was Yakov Pesin, whose involvement was constant and most valuable. And a successful workshop does not by itself lead to written works of interest, so I wish to thank the authors of the articles in this volume for their contributions. Thanks also go to Silvio Levy for his smooth handling of the entire production process, and to Kathleen Hasselblatt for her support.

The aim of the workshop and this volume is to impact the development of dynamical systems, and to that end we paid some attention to making it possible for younger participants to attend the workshop, and the surveys in this volume may attract young mathematicians to those subject areas. The Mathematical Sciences Research Institute adds to the impact of the event by maintaining streaming video of the lectures given at the workshop, which enables everyone to view these lectures (see http://www.msri.org/calendar/workshops/WorkshopInfo/267/show_workshop). Finally, a problem list in this volume is hoped to inspire research into subjects that posed challenges at the time of the workshop. I hope that the readers will find this volume interesting, useful and inspiring.

I am writing these lines on the sixty-second birthday of Anatole Katok and wish to dedicate this volume to him.

Boris Hasselblatt
Somerville (MA), August 2006

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Quantitative symplectic geometry

KAI CIELIEBAK, HELMUT HOFER,
JANKO LATSCHEV, AND FELIX SCHLENK

Dedicated to Anatole Katok on the occasion of his sixtieth birthday

A symplectic manifold (M, ω) is a smooth manifold M endowed with a nondegenerate and closed 2-form ω. By Darboux's Theorem such a manifold looks locally like an open set in some $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with the standard symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j, \tag{0–1}$$

and so symplectic manifolds have no local invariants. This is in sharp contrast to Riemannian manifolds, for which the Riemannian metric admits various curvature invariants. Symplectic manifolds do however admit many global numerical invariants, and prominent among them are the so-called symplectic capacities.

Symplectic capacities were introduced in 1990 by I. Ekeland and H. Hofer [18; 19] (although the first capacity was in fact constructed by M. Gromov [39]). Since then, lots of new capacities have been defined [16; 29; 31; 43; 48; 58; 59; 88; 97] and they were further studied in [1; 2; 8; 9; 25; 20; 27; 30; 34; 36; 37; 40; 41; 42; 45; 47; 49; 51; 55; 56; 57; 60; 61; 62; 63; 65; 71; 72; 73; 86; 87; 89; 90; 92; 95; 96]. Surveys on symplectic capacities are [44; 49; 54; 66; 95]. Different capacities are defined in different ways, and so relations between capacities often lead to surprising relations between different aspects of symplectic geometry and Hamiltonian dynamics. This is illustrated in Section 2, where we discuss some examples of symplectic capacities and describe a few consequences of their existence. In Section 3 we present an attempt to

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better understand the space of all symplectic capacities, and discuss some further general properties of symplectic capacities. In Section 4, we describe several new relations between certain symplectic capacities on ellipsoids and polydiscs. Throughout the discussion we mention many open problems.

As illustrated below, many of the quantitative aspects of symplectic geometry can be formulated in terms of symplectic capacities. Of course there are other numerical invariants of symplectic manifolds which could be included in a discussion of quantitative symplectic geometry, such as the invariants derived from Hofer's bi-invariant metric on the group of Hamiltonian diffeomorphisms, [43; 79; 82], or Gromov–Witten invariants. Their relation to symplectic capacities is not well understood, and we will not discuss them here.

We start out with a brief description of some relations of symplectic geometry to neighboring fields.

1. Symplectic geometry and its neighbors

Symplectic geometry is a rather new and vigorously developing mathematical discipline. The "symplectic explosion" is described in [21]. Examples of symplectic manifolds are open subsets of $(\mathbb{R}^{2n}, \omega_0)$, the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ endowed with the induced symplectic form, surfaces equipped with an area form, Kähler manifolds like complex projective space $\mathbb{CP}^n$ endowed with their Kähler form, and cotangent bundles with their canonical symplectic form. Many more examples are obtained by taking products and through more elaborate constructions, such as the symplectic blow-up operation. A diffeomorphism φ on a symplectic manifold (M, ω) is called *symplectic* or a *symplectomorphism* if $\varphi^*\omega = \omega$.

A fascinating feature of symplectic geometry is that it lies at the crossroad of many other mathematical disciplines. In this section we mention a few examples of such interactions.

Hamiltonian dynamics. Symplectic geometry originated in Hamiltonian dynamics, which originated in celestial mechanics. A time-dependent Hamiltonian function on a symplectic manifold (M, ω) is a smooth function $H\colon \mathbb{R} \times M \to \mathbb{R}$. Since ω is nondegenerate, the equation

$$\omega(X_H, \cdot) = dH(\cdot)$$

defines a time-dependent smooth vector field X_H on M. Under suitable assumption on H, this vector field generates a family of diffeomorphisms φ_H^t called the *Hamiltonian flow* of H. As is easy to see, each map φ_H^t is symplectic. A *Hamiltonian diffeomorphism* φ on M is a diffeomorphism of the form φ_H^1.

Symplectic geometry is the geometry underlying Hamiltonian systems. It turns out that this geometric approach to Hamiltonian systems is very fruitful. Explicit examples are discussed in Section 2 below.

Volume geometry. A volume form Ω on a manifold M is a top-dimensional nowhere vanishing differential form, and a diffeomorphism φ of M is *volume preserving* if $\varphi^*\Omega = \Omega$. Ergodic theory studies the properties of volume preserving mappings. Its findings apply to symplectic mappings. Indeed, since a symplectic form ω is nondegenerate, ω^n is a volume form, which is preserved under symplectomorphisms. In dimension 2 a symplectic form is just a volume form, so that a symplectic mapping is just a volume preserving mapping. In dimensions $2n \geq 4$, however, symplectic mappings are much more special. A geometric example for this is Gromov's Nonsqueezing Theorem stated in Section 2.2 and a dynamical example is the (partly solved) Arnol'd conjecture stating that Hamiltonian diffeomorphisms of closed symplectic manifolds have at least as many fixed points as smooth functions have critical points. For another link between ergodic theory and symplectic geometry see [81].

Contact geometry. Contact geometry originated in geometrical optics. A contact manifold (P, α) is a $(2n-1)$-dimensional manifold P endowed with a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on P. The vector field X on P defined by $d\alpha(X, \cdot) = 0$ and $\alpha(X) = 1$ generates the so-called Reeb flow. The restriction of a time-independent Hamiltonian system to an energy surface can sometimes be realized as the Reeb flow on a contact manifold. Contact manifolds also arise naturally as boundaries of symplectic manifolds. One can study a contact manifold (P, α) by symplectic means by looking at its symplectization $\big(P \times \mathbb{R}, d(e^t \alpha)\big)$, see e.g. [46; 22].

Algebraic geometry. A special class of symplectic manifolds are Kähler manifolds. Such manifolds (and, more generally, complex manifolds) can be studied by looking at holomorphic curves in them. M. Gromov [39] observed that some of the tools used in the Kähler context can be adapted for the study of symplectic manifolds. One part of his pioneering work has grown into what is now called Gromov–Witten theory, see e.g. [70] for an introduction.

Many other techniques and constructions from complex geometry are useful in symplectic geometry. For example, there is a symplectic version of blowing-up, which is intimately related to the symplectic packing problem, see [64; 68] and 4.1.2 below. Another example is Donaldson's construction of symplectic submanifolds [17]. Conversely, symplectic techniques proved useful for studying problems in algebraic geometry such as Nagata's conjecture [5; 6; 68] and degenerations of algebraic varieties [7].

Riemannian and spectral geometry. Recall that the differentiable structure of a smooth manifold M gives rise to a canonical symplectic form on its cotangent bundle T^*M. Giving a Riemannian metric g on M is equivalent to prescribing its unit cosphere bundle $S_g^*M \subset T^*M$, and the restriction of the canonical 1-form from T^*M gives S^*M the structure of a contact manifold. The Reeb flow on S_g^*M is the geodesic flow (free particle motion).

In a somewhat different direction, each symplectic form ω on some manifold M distinguishes the class of Riemannian metrics which are of the form $\omega(J\cdot,\cdot)$ for some almost complex structure J.

These (and other) connections between symplectic and Riemannian geometry are by no means completely explored, and we believe there is still plenty to be discovered here. Here are some examples of known results relating Riemannian and symplectic aspects of geometry.

Lagrangian submanifolds. A middle-dimensional submanifold L of (M,ω) is called *Lagrangian* if ω vanishes on TL.

(i) Volume. Endow complex projective space $\mathbb{CP}^n$ with the usual Kähler metric and the usual Kähler form. The volume of submanifolds is taken with respect to this Riemannian metric. According to a result of Givental–Kleiner–Oh, the standard $\mathbb{RP}^n$ in $\mathbb{CP}^n$ has minimal volume among all its Hamiltonian deformations [74]. A partial result for the Clifford torus in $\mathbb{CP}^n$ can be found in [38]. The torus $S^1 \times S^1 \subset S^2 \times S^2$ formed by the equators is also volume minimizing among its Hamiltonian deformations, [50]. If L is a closed Lagrangian submanifold of $(\mathbb{R}^{2n},\omega_0)$, there exists according to [98] a constant C depending on L such that

$$\operatorname{vol}(\varphi_H(L)) \geq C \quad \text{for all Hamiltonian deformations of } L. \tag{1–1}$$

(ii) Mean curvature. The mean curvature form of a Lagrangian submanifold L in a Kähler–Einstein manifold can be expressed through symplectic invariants of L, see [15].

The first eigenvalue of the Laplacian. Symplectic methods can be used to estimate the first eigenvalue of the Laplace operator on functions for certain Riemannian manifolds [80].

Short billiard trajectories. Consider a bounded domain $U \subset \mathbb{R}^n$ with smooth boundary. There exists a periodic billiard trajectory on $\overline{U}$ of length l with

$$l^n \leq C_n \operatorname{vol}(U) \tag{1–2}$$

where C_n is an explicit constant depending only on n, see [98; 30].

2. Examples of symplectic capacities

In this section we give the formal definition of symplectic capacities, and discuss a number of examples along with sample applications.

2.1. Definition. Denote by $Symp^{2n}$ the category of all symplectic manifolds of dimension $2n$, with symplectic embeddings as morphisms. A *symplectic category* is a subcategory $\mathcal{C}$ of $Symp^{2n}$ such that $(M,\omega)\in\mathcal{C}$ implies $(M,\alpha\omega)\in\mathcal{C}$ for all $\alpha>0$.

CONVENTION. *We will use the symbol* $\hookrightarrow$ *to denote symplectic embeddings and* $\to$ *to denote morphisms in the category* $\mathcal{C}$ *(which may be more restrictive).*

Let $B^{2n}(r^2)$ be the open ball of radius r in $\mathbb{R}^{2n}$ and $Z^{2n}(r^2)=B^2(r^2)\times\mathbb{R}^{2n-2}$ the open cylinder (the reason for this notation will become apparent below). Unless stated otherwise, open subsets of $\mathbb{R}^{2n}$ are always equipped with the canonical symplectic form $\omega_0=\sum_{j=1}^n dy_j\wedge dx_j$. We will suppress the dimension $2n$ when it is clear from the context and abbreviate

$$B := B^{2n}(1), \qquad Z := Z^{2n}(1).$$

Now let $\mathcal{C}\subset Symp^{2n}$ be a symplectic category containing the ball B and the cylinder Z. A *symplectic capacity* on $\mathcal{C}$ is a covariant functor c from $\mathcal{C}$ to the category $([0,\infty],\leq)$ (with $a\leq b$ as morphisms) satisfying

(MONOTONICITY): $c(M,\omega)\leq c(M',\omega')$ if there exists a morphism $(M,\omega)\to(M',\omega')$;
(CONFORMALITY): $c(M,\alpha\omega)=\alpha\, c(M,\omega)$ for $\alpha>0$;
(NONTRIVIALITY): $0<c(B)$ and $c(Z)<\infty$.

Note that the (Monotonicity) axiom just states the functoriality of c. A symplectic capacity is said to be *normalized* if

(NORMALIZATION): $c(B)=1$.

As a frequent example we will use the set Op^{2n} of open subsets in $\mathbb{R}^{2n}$. We make it into a symplectic category by identifying $(U,\alpha^2\omega_0)$ with the symplectomorphic manifold $(\alpha U,\omega_0)$ for $U\subset\mathbb{R}^{2n}$ and $\alpha>0$. We agree that the morphisms in this category shall be symplectic embeddings induced by *global* symplectomorphisms of $\mathbb{R}^{2n}$. With this identification, the (Conformality) axiom above takes the form

(CONFORMALITY)′: $c(\alpha U)=\alpha^2 c(U)$ for $U\in Op^{2n}$, $\alpha>0$.

2.2. Gromov radius. In view of Darboux's Theorem one can associate with each symplectic manifold (M,ω) the numerical invariant

$$c_B(M,\omega) := \sup\{\alpha > 0 \mid B^{2n}(\alpha) \hookrightarrow (M,\omega)\}$$

called the *Gromov radius* of (M,ω), [39]. It measures the symplectic size of (M,ω) in a geometric way, and is reminiscent of the injectivity radius of a Riemannian manifold. Note that it clearly satisfies the (Monotonicity) and (Conformality) axioms for a symplectic capacity. It is equally obvious that $c_B(B) = 1$.

If M is 2-dimensional and connected, then $\pi c_B(M,\omega) = \int_M \omega$, i.e. c_B is proportional to the volume of M, see [89]. The following theorem from Gromov's seminal paper [39] implies that in higher dimensions the Gromov radius is an invariant very different from the volume.

NONSQUEEZING THEOREM (GROMOV, 1985). *The cylinder $Z \in Symp^{2n}$ satisfies $c_B(Z) = 1$.*

Therefore the Gromov radius is a normalized symplectic capacity on $Symp^{2n}$. Gromov originally obtained this result by studying properties of moduli spaces of pseudo-holomorphic curves in symplectic manifolds.

It is important to realize that the existence of at least one capacity c with $c(B) = c(Z)$ also *implies* the Nonsqueezing Theorem. We will see below that each of the other important techniques in symplectic geometry (such as variational methods and the global theory of generating functions) gave rise to the construction of such a capacity, and hence an independent proof of this fundamental result.

It was noted in [18] that the following result, originally established by Eliashberg and by Gromov using different methods, is also an easy consequence of the existence of a symplectic capacity.

THEOREM (ELIASHBERG, GROMOV). *The group of symplectomorphisms of a symplectic manifold (M,ω) is closed for the compact-open C^0-topology in the group of all diffeomorphisms of M.*

2.3. Symplectic capacities via Hamiltonian systems. The next four examples of symplectic capacities are constructed via Hamiltonian systems. A crucial role in the definition or the construction of these capacities is played by the action functional of classical mechanics. For simplicity, we assume that $(M,\omega) = (\mathbb{R}^{2n},\omega_0)$. Given a Hamiltonian function $H\colon S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ which is periodic in the time-variable $t \in S^1 = \mathbb{R}/\mathbb{Z}$ and which generates a global flow φ_H^t, the action functional on the loop space $C^\infty(S^1, \mathbb{R}^{2n})$ is defined as

$$\mathcal{A}_H(\gamma) = \int_\gamma y\,dx - \int_0^1 H\big(t, \gamma(t)\big)dt. \tag{2–1}$$

Its critical points are exactly the 1-periodic orbits of φ_H^t. Since the action functional is neither bounded from above nor from below, critical points are saddle points. In his pioneering work [83; 84], P. Rabinowitz designed special minimax principles adapted to the hyperbolic structure of the action functional to find such critical points. We give a heuristic argument why this works. Consider the space of loops

$$E = H^{1/2}(S^1, \mathbb{R}^{2n}) = \left\{ z \in L^2\left(S^1; \mathbb{R}^{2n}\right) \;\middle|\; \sum_{k\in\mathbb{Z}} |k|\,|z_k|^2 < \infty \right\}$$

where $z = \sum_{k\in\mathbb{Z}} e^{2\pi k t J} z_k$, $z_k \in \mathbb{R}^{2n}$, is the Fourier series of z and J is the standard complex structure of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The space E is a Hilbert space with inner product

$$\langle z, w\rangle = \langle z_0, w_0\rangle + 2\pi \sum_{k\in\mathbb{Z}} |k|\,\langle z_k, w_k\rangle,$$

and there is an orthogonal splitting $E = E^- \oplus E^0 \oplus E^+$, $z = z^- + z^0 + z^+$, into the spaces of $z \in E$ having nonzero Fourier coefficients $z_k \in \mathbb{R}^{2n}$ only for $k < 0$, $k = 0$, $k > 0$. The action functional $\mathcal{A}_H\colon C^\infty(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ extends to E as

$$\mathcal{A}_H(z) = \left(\tfrac{1}{2}\left\|z^+\right\|^2 - \tfrac{1}{2}\left\|z^-\right\|^2\right) - \int_0^1 H(t, z(t))\,dt. \tag{2–2}$$

Notice now the hyperbolic structure of the first term $\mathcal{A}_0(x)$, and that the second term is of lower order. Some of the critical points $z(t) \equiv const$ of $\mathcal{A}_0$ should thus persist for $H \neq 0$.

2.3.1. Ekeland–Hofer capacities. The first construction of symplectic capacities via Hamiltonian systems was carried out by Ekeland and Hofer [18; 19]. To give the heuristics, we consider a bounded domain $U \subset \mathbb{R}^{2n}$ with smooth boundary ∂U. A *closed characteristic* γ on ∂U is an embedded circle in ∂U tangent to the characteristic line bundle

$$\mathcal{L}_U = \{(x, \xi) \in T\partial U \mid \omega_0(\xi, \eta) = 0 \text{ for all } \eta \in T_x\partial U\}.$$

If ∂U is represented as a regular energy surface $\left\{x \in \mathbb{R}^{2n} \mid H(x) = \text{const}\right\}$ of a smooth function H on $\mathbb{R}^{2n}$, then the Hamiltonian vector field X_H restricted to ∂U is a section of $\mathcal{L}_U$, and so the traces of the periodic orbits of X_H on ∂U are the closed characteristics on ∂U. The *action* of a closed characteristic γ on ∂U is defined as $\mathcal{A}(\gamma) = \left| \int_\gamma y\, dx \right|$. The set

$$\Sigma(U) = \{k\, \mathcal{A}(\gamma) \mid k = 1, 2, \dots;\ \gamma \text{ is a closed characteristic on } \partial U\}$$

is called the *action spectrum* of U. Now one would like to associate with U suitable elements of $\Sigma(U)$. Without further assumptions on U, however, the set $\Sigma(U)$ may be empty (see [32; 33; 35]), and there is no obvious way to achieve (Monotonicity). To salvage this naive idea, Ekeland and Hofer considered for each bounded open subset U of $\mathbb{R}^{2n}$ the space $\mathcal{F}(U)$ of time-independent Hamiltonian functions $H\colon \mathbb{R}^{2n} \to [0, \infty)$ satisfying

- $H \equiv 0$ on some open neighbourhood of $\overline{U}$, and
- $H(z) = a|z|^2$ for $|z|$ large, where $a > \pi$, $a \notin \mathbb{N}\pi$.

Notice that the circle S^1 acts on the Hilbert space E by time-shift $x(t) \mapsto x(t+\theta)$ for $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. The special form of $H \in \mathcal{F}(U)$ guarantees that for each $k \in \mathbb{N}$ the equivariant minimax value

$$c_{H,k} := \inf \left\{ \sup_{\gamma \in \xi} \mathcal{A}_H(\gamma) \mid \xi \subset E \text{ is } S^1\text{-equivariant and } \operatorname{ind}(\xi) \geq k \right\}$$

is a critical value of the action functional (2–2). Here, $\operatorname{ind}(\xi)$ denotes a suitable Fadell–Rabinowitz index [26; 19] of the intersection $\xi \cap S^+$ of ξ with the unit sphere $S^+ \subset E^+$. The *k-th Ekeland–Hofer capacity* c_k^{EH} on the symplectic category $\mathcal{O}p^{2n}$ is now defined as

$$c_k^{\mathrm{EH}}(U) := \inf \left\{ c_{H,k} \mid H \in \mathcal{F}(U) \right\}$$

if $U \subset \mathbb{R}^{2n}$ is bounded and as

$$c_k^{\mathrm{EH}}(U) := \sup \left\{ c_k^{\mathrm{EH}}(V) \mid V \subset U \text{ bounded} \right\}$$

in general. It turns out that these numbers are indeed symplectic capacities. Moreover, they realize the naive idea of picking out suitable elements of $\Sigma(U)$ for many U: A bounded open subset U of $\mathbb{R}^{2n}$ is said to be of *restricted contact type* if its boundary ∂U is smooth and if there exists a vector field v on $\mathbb{R}^{2n}$ which is transverse to ∂U and whose Lie derivative satisfies $L_v\omega_0 = \omega_0$. Examples are bounded star-shaped domains with smooth boundary.

PROPOSITION (EKELAND AND HOFER, 1990). *If U is of restricted contact type, then $c_k^{\mathrm{EH}}(U) \in \Sigma(U)$ for each $k \in \mathbb{N}$.*