Non-classical Vibrations of Arches and Beams

Eigenvalues and Eigenfunctions

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Preface

Deformable systems (DS) with distributed parameters such as beams and arches, are widely used in modern engineering. These structures find wide applications in civil and transport engineering, in mechanical, robotics and radio-engineering (load-bearing members, electric drives for robotics and mechanisms, boards of a radio-electronic apparatus), etc. The common fundamental feature of these structures is that their dynamical behavior is described by partial differential equations.

Analysis of any vibration problem of DS starts from definition of its fundamental characteristics, such as eigenvalues and eigenfunctions. For their calculation, there exist modern analytical and numerical methods, which are applicable for different mathematical models of DS. These models describe specific effects and operating conditions.

With the development of high technologies, the purpose of DS and their functional peculiarities as part of an engineering system in whole is changed. Also, the operating conditions of the structures are changed. For example, elastic elements become objects of active control; elastic beam elements are used as mechanical filters in electronics; elastic DS are used in control and measurement systems, which include elements of different natures, such as electrical, acoustical, optical, magnetic; beam systems are widely used as resonant strain gauges in micro-mechanical systems for the measurement of forces, accelerations, displacements and pressure. There also exist multi-purpose mechanical systems where elastic elements act simultaneously as load-bearing elements and functional devices, for a special purpose. Of course, the list of engineering fields where elastic DS are used is much wider than presented above.

The higher demands to a dynamical structure in whole leads to increasing demands of each part of a structure and in particular, to its DS. It means that the requirements to the accuracy of calculation of eigenvalues and eigenfunctions of DS are increased. However, on the basis of the simplified mathematical models it is impossible to obtain refined fundamental characteristics and take into account the specific effects.

This handbook presents solutions of eigenvalues and eigenfunctions for the advanced analysis of beams and arches. Analysis of beams on the basis of mathematical models, which take into account different additional effects, such as the effects of rotary inertia and shear force, are considered. Analysis of DS with specific conditions of a structure in service, such as elastic foundation, axial tensile or compressive load, is shown. Also, the handbook presents different types of nonlinear vibration problems of the beams, some results dealing with vibration of optimal designed beams. Special attention is paid to eigenvalue and eigenfunction problems for arches with different boundary conditions and with non-uniform cross-section.

During the last thirty years, a vast amount of information dealing with eigenvalues and eigenfunctions of DS has been accumulated. However, this information is spread out over numerous articles that are published in journals, conference proceedings, guidelines, departmental reports and theses. Existing handbooks do not reflect, in a reasonable manner, this important problem. For practicing engineers and researchers at universities and institutions, searching the vast literature, even with ready access to computerized databases and the Internet for a specific type of problem, this is a difficult and time consuming task. Solutions of many important problems remain unknown to specialists, who could greatly benefit from such knowledge.

The objective of this handbook is to provide the most comprehensive, up-to-date reference of known solutions to a large variety of vibration problems of beams and arches. The intent is to provide information that is not available in current handbooks and to provide solutions for the eigenvalues and eigenfunctions problems that engineers and researchers use for the advanced analysis of dynamical behavior of beams and arches.

The most distinctive feature of this handbook is that it is the most complete collection of eigenvalues and eigenfunctions for different types of beams and arches that has ever been published. It includes a large number of cases of beams and arches with concentrated and distributed parameters with different types of elastic supports and boundary conditions. All problems in this book may be considered as non-classical problems. Authors understand that division of vibration of deformable structures as classical and non-classical is conventional. However, this division is convenient in order to consider non-classical problems separately, i.e., such problems, which take into account additional important effects (shear, rotary inertia, etc), consider nonlinear problems by analytical methods (static, physical, geometrical non-linearity), problems of optimal design, as well as some special problems, for example vibration of beams in magnetic field. Of course, this list of non-classical problems is not complete. The authors have conducted a very extensive research of published materials in many countries and compiled solutions to different cases of vibration of deformable systems. The criteria for the selection of problems included in the handbook were mainly based on the importance and the frequency of appearance of the problem in practical engineering applications. Problems selection is based on the 35 years' combined experience of the authors in the field of structural dynamics.

To compile the information presented in this handbook, the authors carefully reviewed monographs, journals, handbooks, proceedings, preprints and theses, as well as results of the authors' own research. The handbook contains the fundamental and most up-to-date results concerning eigenvalues and eigenfunctions of beams and arches. The majority of the sources consulted have been published in the USA, Canada, England, Russia, Germany, Japan, Israel, and Netherlands over the past 40 years. Each case presented in the handbook is properly referenced. The majority of the results, which are presented in the original sources, have been independently verified by the authors.

Authors will appreciate comments and suggestions to improve the current edition. All constructive criticism will be accepted with gratitude.

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Notation

x, y, z	Cartesian coordinates
x	Spatial coordinate
t	Time
E, G	Young's and shear modulus of the beam material, $2G(1+\nu) = E$
ν, ρ	Poisson's ratio and density of the beam material
σ, ε	Stress and strain of material of a beam
σ_{xx}, σ_{xy}	Longitudinal and shear stresses
u_x, u_y	Longitudinal and transversal displacements of a rod
c_t, c_b	Velocity of shear and longitudinal waves, $c_t^2\rho = G$; $c_b^2\rho = E$
$k_b k_t$	Longitudinal and shear propagation constant, $k_b = \omega/c_b$, $k_t = \omega/c_t$
k_0	Bending wave number for Bernoulli-Euler rod, $k_0^4 D_0^4 = \omega^2$
l	Length of the beam
h, d, b	Geometrical dimensional of cross sectional of the beam
I_n	Cross sectional area moment of inertia of order n
I, I_2	Second moment of inertia of a cross-section area with respect to neutral line
m	Mass per unit length of the beam, $m = \rho A$
M, J	Lumped mass and moment inertia of the mass
k_{tr}, k_{rot}	Translational and rotational stiffness coefficients
k_{tr}^*, k_{rot}^*	Dimensionless stiffness coefficients, $k_{tr}^* EI = k_{tr} l^3$, $k_{rot}^* EI = k_{rot} l$
$A(x)$	Cross-sectional area of a beam
EI, i	Bending stiffness and bending stiffness per unit length, $i = EI/l$
$y(x, t), \psi(x, t)$	Lateral displacement and slope of the beam
E_F, ρ_F	Young's modulus and density of the foundation material
G_0	Foundation modulus of rigidity (Pasternak model)
k_{tr}, k_0	Elastic transverse translatory stiffness of medium (Winkler foundation modulus)
k_{slope}, D_0	Elastic sloping stiffness of medium
k_{tilt}	Elastic tilting (transverse rotating) stiffness of medium, Nm/m
T	Axial load
U	Dimensionless parameter, $2UEI = Tl^2$
T_E	First Euler critical load
M, Q	Bending moment and shear force
U, T	Potential and kinetic energy

$X(x), \psi(x)$	Mode shapes
k	Shear factor
λ, k	Frequency parameters, $\lambda^4 EI = ml^4\omega^2$, $k^4EI = m\omega^2$, $\lambda = kl$
ω_{0i}	Frequency of transverse vibration of a beam no axial force in ith mode vibration
Ω_{0i}	Dimensionless frequency parameter of beam no axial force in ith mode vibration; $\Omega_{0i}\alpha = \omega_{0i}l^2$, $\alpha^2 = EI$, $\Omega_{0i} = \lambda^2$
ω	Frequency of transverse vibration compressed beam (relative natural frequency);
Ω	Dimensionless frequency parameter of compressed beam (relative natural frequency); $\Omega\alpha = \omega l^2$
$\Omega^* = \Omega/\Omega_{0i}$	Normalized natural frequency parameter
H	Hamiltonian
k_1, k_2	Lagrange multipliers
V	Volume of a beam
V_-, V_+	Lower and upper limit of the volume of a beam
ω^-, ω^+	Lower and upper bounds of the frequency vibration
G	Gauge factor
v, v_{cr}	Velocity and critical velocity of the moving liquid
ξ, r	Dimensionless coordinate and radius of gyration, $\xi = x/l$, $0 \le \xi \le 1$, $r^2Al^2 = I$
s, ζ	Dimensionless parameters, $s^2kAGl^2 = EI$
R	Correction factor
k	Parameter of magnetic field
β, β_F	Nonlinear parameter of the beam and foundation
P	Internal pressure
s	Stiffness coefficients
$s(M\psi)$	Flexural stiffness coefficient of moment due to rotational deformation
$s(MX)$	Flexural stiffness coefficient of moment due to transverse deformation
$s(V\psi)$	Flexural stiffness coefficient of shear due to rotational deformation
$s(VX)$	Flexural stiffness coefficient of shear due to transverse deformation
V_i	Puzyrevsky functions
w, v	Radial and tangential displacements of an arch
$R(\alpha), f, l$	Radius of curvature, rise and span of an arch
$(\bullet), (')$	Differentiation with respect to time and space coordinate

NON-CLASSICAL VIBRATIONS OF ARCHES AND BEAMS

Contents

CHAPTER 1

TRANSVERSE VIBRATION EQUATIONS

The different assumptions and corresponding theories of transverse vibrations of beams are presented. The dispersive equation, its corresponding curve 'propagation constant–frequency' and its comparison with the exact dispersive curve are presented for each theory and discussed.

The exact dispersive curve corresponds to the first and second antisymmetrical Lamb's wave.

1.1 AVERAGE VALUES AND RESOLVING EQUATIONS

The different theories of dynamic behaviours of beams may be obtained from the equations of the theory of elasticity, which are presented with respect to average values. The object under study is a thin plate with rectangular cross-section (Figure 1.1).

1.1.1 Average values for deflections and internal forces

1. Average displacement and slope are

$$w = \int_{-H}^{+H} \frac{u_y}{2H} \mathrm{d}y \tag{1.1}$$

$$\psi = \int_{-H}^{+H} \frac{y u_x}{I_z} \mathrm{d}y \tag{1.2}$$

where u_x and u_y are longitudinal and transverse displacements.

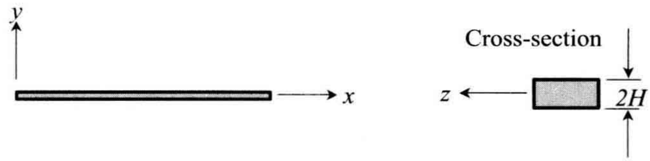

FIGURE 1.1. Thin rectangular plate, the boundary conditions are not shown.

2. Shear force and bending moment are

$$Q = \int_{-H}^{+H} \sigma_{xy}\, \mathrm{d}y \tag{1.3}$$

$$M = \int_{-H}^{+H} y\sigma_{xx}\, \mathrm{d}y \tag{1.4}$$

where σ_x and σ_y are the normal and shear stresses that correspond to u_x and u_y.

Resolving the equations may be presented in terms of average values as follows (Landau and Lifshitz, 1986)

1. Integrating the equilibrium equation of elasticity theory leads to

$$2\rho H\ddot{w} = Q' \tag{1.5}$$

$$\rho I_z\ddot{\psi} = M_z' - Q \tag{1.6}$$

2. Integrating Hooke's equation for the plane stress leads to

$$Q = 2HG\left[w' + \frac{u_x(H)}{H}\right] \tag{1.7}$$

$$M_z = E_1\{I_z\psi' + 2H\nu[u_y(H) - w]\} = EI_z\psi' + \nu\int_{-H}^{+H} y\sigma_{yy}\, \mathrm{d}y \tag{1.7a}$$

Equations (1.5)–(1.7a) are complete systems of equations of the theory of elasticity with respect to average values w, ψ, Q and M. These equations contain two redundant unknowns $u_x(H)$ and $u_y(H)$. Thus, to resolve the above system of equations, additional equations are required. These additional equations may be obtained from the assumptions accepted in approximate theories.

The solution of the governing differential equation is

$$w = \exp(ikx - i\omega t) \tag{1.8}$$

where k is a propagation constant of the wave and ω is the frequency of vibration.

The degree of accuracy of the theory may be evaluated by a dispersive curve $k - \omega$ and its comparison with the exact dispersive curve. We assume that the exact dispersive curve is one that corresponds to the first and second antisymmetric Lamb's wave. The closer the dispersive curve for a specific theory to the exact dispersive curve, the better the theory describes the vibration process. Short analysis of equations for transversal vibrations on the basis of different theories are shown below. (Artobolevsky *et al.* 1979).

1.2 FUNDAMENTAL THEORIES AND APPROACHES

1.2.1 Bernoulli–Euler theory

The Bernoulli–Euler theory takes into account the inertia forces due to the transverse translation and neglects the effect of shear deflection and rotary inertia.

Assumptions

1. The cross-sections remain plane and orthogonal to the neutral axis ($\psi = -w'$).
2. The longitudinal fibres do not compress each other ($\sigma_{yy} = 0, \to M_z = EI_z\psi'$).
3. The rotational inertia is neglected ($\rho I_z\ddot{\psi} = 0$). This assumption leads to

$$Q = -M_z' = -EI_z w'''$$

Substitution of the previous expression in Equation (1.5) leads to the differential equation describing the transverse vibration of the beam

$$\frac{\partial^4 w}{\partial x^4} + \frac{1}{D_0^4}\frac{\partial^2 w}{\partial t^2} = 0, \quad D_0^4 = \frac{EI_z}{2\rho H} \tag{1.9}$$

Let us assume that displacement w is changed according to Equation (1.8). The dispersive equation which establishes the relationship between k and ω may be presented as

$$k^4 = \frac{\omega^2}{D_0^4} = k_0^4$$

This equation has two roots for a forward-moving wave in a beam and two roots for a backward-moving wave. Positive roots correspond to a forward-moving wave, while negative roots correspond to a backward-moving wave.

The results of the dispersive relationships are shown in Figure 1.2; dimensionless parameters are $\lambda = kH$, $\mu_t = k_t H$. Here, bold curves 1 and 2 represent the exact results. Curves 1 and 2 correspond to the first and second antisymmetric Lamb's wave, respectively. The second wave transfers from the imaginary zone into the real one at $k_t H = \pi/2$. Curves 3 and 4 are in accordance with the Bernoulli–Euler theory. Dispersion obtained from this theory and dispersion obtained from the exact theory give a close result when frequencies are close to zero. This elementary beam theory is valid only when the height of the beam is small compared with its length (Artobolevsky *et al.*, 1979).

1.2.2 Rayleigh theory

This theory takes into account the effect of rotary inertia (Rayleigh, 1877).

Assumptions

1. The cross-sections remain plane and orthogonal to the neutral axis ($\psi = -w'$).
2. The longitudinal fibres do not compress each other ($\sigma_{yy} = 0$, $M_z = EI_z\psi'$).

From Equation (1.6) the shear force $Q = M_z' - \rho I_z\ddot{\psi}$.

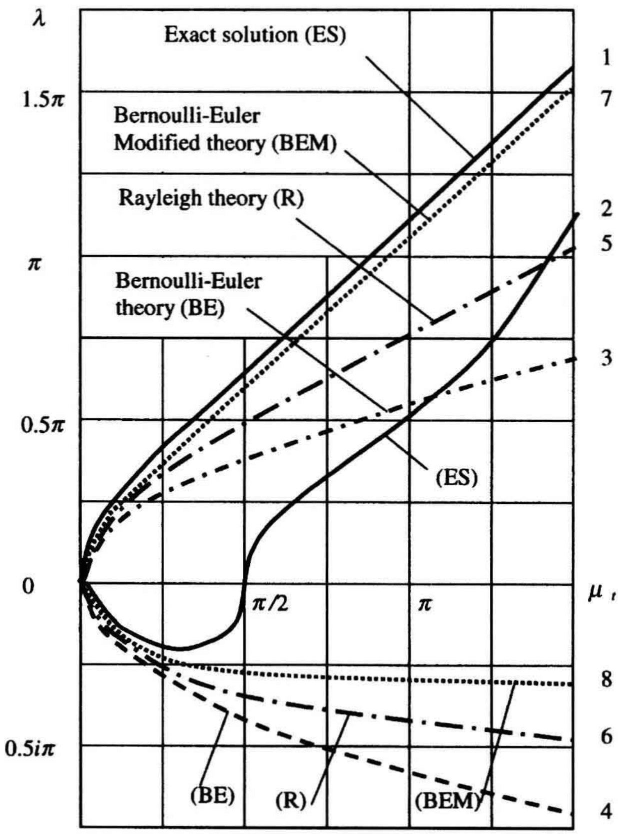

FIGURE 1.2. Transverse vibration of beams. Dispersive curves for different theories. 1, 2–Exact solution; 3, 4–Bernoulli–Euler theory; 5, 6–Rayleigh theory, 7, 8–Bernoulli–Euler modified theory.

Differential equation of transverse vibration of the beam

$$\frac{\partial^4 w}{\partial x^4}+\frac{1}{D_0^4}\frac{\partial^2 w}{\partial t^2}-\frac{1}{c_b^2}\frac{\partial^4 w}{\partial x^2\,\partial t^2}=0, \quad c_b^2=\frac{E}{\rho} \tag{1.10}$$

where c_b is the velocity of longitudinal waves in the thin rod.

The last term on the left-hand side of the differential equation describes the effect of the rotary inertia.

The dispersive equation may be presented as follows

$$2k_{1,2}^2=k_b^2\pm\sqrt{k_b^2+4k_0^4}, \quad k_b^2=\frac{\omega^2}{c_b^2}$$

where k_0 is the wave number for the Bernoulli–Euler rod, and k_b is the longitudinal wave number.

Curves 5 and 6 in Figure 1 reflect the effect of rotary inertia.