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Lifting Modules

Supplements and Projectivity in Module Theory

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Frontiers in Mathematics

Preface

Extending modules are generalizations of injective modules and, dually, *lifting modules* generalise projective supplemented modules. However, while every module has an injective hull it does not necessarily have a projective cover. This creates a certain asymmetry in the duality between extending modules and lifting modules; it stems from the fact that in any module M there exist, by Zorn's Lemma, (intersection) complements for any submodule N but, by contrast, supplements for N in M need not exist. (A module is *extending*, or *CS*, if every complement submodule is a direct summand, and it is *lifting* if it is amply supplemented and every supplement is a direct summand.) The terms *extending* and *lifting* were coined by Harada and Oshiro (e.g., [156, 272]).

The monograph *Continuous and Discrete Modules* by Mohamed and Müller [241] considers both extending and lifting modules while the subsequent *Extending Modules* [85] presents a deep insight into the structure of the former class.

The purpose of our monograph is to provide a similarly up-to-date account of lifting modules. However, this involves more than a routine dualisation of the results on extending modules in [85] since, as alluded to in our opening paragraph, it becomes necessary to be mindful of the existence of supplements and their properties. An investigation of supplements leads naturally to the topics of *semilocal modules* and *hollow* (*dual Goldie*) *dimension*, as studied in [56, 226, 228].

A number of results concerning lifting modules have appeared in the literature in recent years. Major contributions to their theory came, for example, from research groups in Japan, Turkey, and India. Moreover, progress in other branches of module theory has also had a flow-on effect, providing further enrichment to lifting module theory. Many of the sections in our text end with comments which highlight the development of the concepts and results presented and give further details of the contributors and their work.

One aspect of these investigations is the (co)torsion-theoretic point of view as elaborated in, for example, [329]. This provides a way of studying the corresponding notions in the category of comodules over coalgebras (or corings) which behaves similarly to the category $\sigma[M]$ of modules subgenerated by some given module M.

Roughly speaking, the various classes of modules associated with extending modules are obtained by weakening the injectivity properties correspondingly. Consequently, our dual concept of lifting modules is pursued through relaxed projectivity conditions. However, since the existence of supplements in a module M is not an immediate consequence of (weak) projectivity conditions, we must frequently rely also on either finiteness conditions or structural conditions on the lattice of submodules of M (such as the AB5* condition).

The authors gratefully acknowledge the contributions of many colleagues and friends to the present monograph. A considerable part of the material consists of

reports on their results and some effort was made to coordinate this information and profit from the induced synergy.

They also want to express their thanks for the financial support behind the project. In particular, Christian Lomp was partially supported by the Centro de Matemática da Universidade do Porto, financed by FCT (Portugal) through the programs POCTI and POSI, with national and European Community structural funds. The cooperation between the partners in Germany and India was supported by the German Research Foundation (DFG) and the German Academic Exchange Service (DAAD).

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Introduction

In the first chapter, we present basic notions and techniques from module theory which are employed in the rest of the text. While proofs are often provided, the reader is referred to standard texts for the background details for the more common concepts. Section 1 considers the fundamentals of injectivity, setting the scene for the dual concepts to be introduced later. Similarly, the reader is reminded here of independent families of submodules, prior to the dual notion of coindependence. Small submodules, the radical, hollow modules, local modules, and max modules are considered in Section 2. The next section sees the introduction of coclosed submodules and coclosure (both key ingredients of Section 20 onwards) while Section 4 looks at an assortment of projectivity conditions and how these can influence the endomorphism ring of a module. These weaker forms of projectivity play important rôles throughout the rest of the monograph. The chapter ends with a section on the hollow dimension of a module, this being the dual of the more familiar Goldie or uniform dimension.

Chapter 2 considers torsion-theoretic aspects, the first two sections looking at preradicals, colocalisation, and torsion theories associated with small submodules and projectivity. Section 8 introduces small modules. Here, given two modules M and N over the ring R, we say that N is *M-small* if N is small in some L in $\sigma[M]$. The preradical and torsion theories (co)generated by the class of M-small modules are examined in detail. Corational modules and in particular minimal corational submodules make their appearance in Section 9, leading to the definition of a copolyform module — these are duals of notions used in the more familiar construction and theory of rings and modules of quotients. The last section of this chapter looks briefly at the theory of proper classes of short exact sequences in $\sigma[M]$ and supplement submodules relative to a radical for $\sigma[M]$, the hope being that in the future these general notions will provide further insight to the theory of lifting modules.

Much of module theory is concerned with the presence (or absence) of decomposability properties and Chapter 3 presents a treatment of those properties which are particularly relevant to the theory of lifting modules. This begins with a section on the exchange property, including characterizations of exchange rings, exchangeable decompositions and the (more general) internal exchange property. This is followed by Section 12 which initially looks at LE-modules, that is, modules with a local endomorphism ring, and modules which decompose as direct sums of LE-modules. After an overview of the background to Azumaya's generalization of the Krull–Schmidt Theorem, this section then looks at Harada's work on the interplay between the exchange property and local semi-T-nilpotency. The main result presented here is an important characterization of the exchange property for a module with an indecomposable decomposition due to Zimmermann-Huisgen and Zimmermann (following earlier results by Harada).

Section 13 introduces the concept of local summands, showing how this is intimately related to local semi-T-nilpotency, in particular through a theorem of Dung which also generalises earlier work by Harada. The next section defines partially invertible homomorphisms and the total, concepts pioneered by Kasch. The coincidence of the Jacobson radical and the total for the endomorphism ring of a module with an LE-decomposition provides yet another link with local semi-T-nilpotency and the exchange property. Section 15 considers rings of stable range 1, unit-regular rings and the cancellation, substitution, and internal cancellation properties — these concepts are inter-related for exchange rings by a result of Yu. Chapter 3 ends with a section on decomposition uniqueness, in particular results of a Krull–Schmidt nature due to Facchini and coauthors, and biuniform modules (modules which are both uniform and hollow).

A *complement* of a submodule N of a module M can be defined to be any submodule L of M maximal with respect to the property that $L \cap N = 0$. Dually, a *supplement* K of N in M can be defined to be any submodule K of M minimal with respect to the property that $K + N = M$. Chapter 4 looks at supplements in modules, a keystone concept for the rest of the monograph. As emphasised in our Preface, in contrast to complements of submodules in a module, supplements of submodules need not exist (consider the $\mathbb{Z}$-module $\mathbb{Z}$, for example). The first section of the chapter introduces and investigates the class of semilocal modules and, more specifically, semilocal rings. Weak supplements are also introduced here and, among other things, weakly supplemented modules (in which every submodule has a weak supplement) are shown to be semilocal (and so, in spite of their more relaxed definition, weak supplements also do not always exist). Section 18 looks at the connection between finite hollow dimension and the existence of weak supplements; in particular, it is shown that the rings over which every finitely generated left (or right) module has finite hollow dimension are those for which every finitely generated module is weakly supplemented and these are precisely the semilocal rings. Modules having semilocal endomorphism rings are investigated in Section 19 and shown to have the n-th root uniqueness property.

Section 20 formally introduces supplements and the classes of supplemented, finitely supplemented, cofinitely supplemented, and amply supplemented modules; the relationships between these classes and hollow submodules and hollow dimension also receive attention. As with their essential closure counterpart, the coclosure (if it exists) of a submodule in a module need not be unique. Modules for which every submodule has a unique coclosure are called *UCC* modules and investigated in the last section of Chapter 4. In particular, as a dual to the relationship between polyform modules and modules with unique complements, an amply supplemented copolyform module M is UCC, and, if $M \oplus M$ is a UCC weakly supplemented module, then M is copolyform and $M \oplus M$ is amply supplemented.

The opening section of the final Chapter 5 introduces the modules of our title, lifting modules. As already mentioned, these are the duals of extending modules. More specifically, a module M is *lifting* if, given any submodule N of M, N lies over a direct summand of M, that is, there is a direct summand X of M such that $X \subseteq N$ and N/X is small in M/X. The building blocks of this class, in other words the indecomposable lifting modules, are precisely the hollow modules. Indeed, any lifting module of finite uniform or finite hollow dimension is a finite direct sum of hollow modules and several other conditions are established that guarantee an indecomposable decomposition for a lifting module. A weaker version of lifting, namely hollow-lifting, is also considered, as well as the effect of chain conditions on the lifting property.

The direct sum of two lifting modules need not be lifting, so Section 23 looks at when the direct sum of finitely many lifting modules is also lifting. The results here involve projectivity conditions defined in Section 4 and exchangeable decompositions from Section 11. Section 24 looks at the lifting property for infinite direct sums, in particular for modules with LE-decompositions (with material from Chapter 3 used extensively) and for direct sums of hollow modules. A module M is said to be Σ-lifting if the direct sum of any family of copies of M is lifting. If M is a Σ-lifting LE-module then it is local and self-projective. Section 25 investigates Σ-lifting modules M and their indecomposable summands, in particular when M is also copolyform or self-injective.

The fairly long Section 26 looks at four subclasses of lifting modules, namely the weakly discrete, quasi-discrete, discrete, and strongly discrete modules (and we have the strict hierarchy: lifting $\Rightarrow$ weakly discrete $\Rightarrow$ quasi-discrete $\Rightarrow$ discrete $\Rightarrow$ strongly discrete). Roughly speaking, each of these may be defined by requiring the module to be supplemented, its supplement submodules to be further constrained, and certain relative projectivity conditions to be satisfied. Each class is characterized and examined in depth, in particular with regard to decomposition, factor modules, and stability under direct sums. The section ends with a look at two particular types of strongly discrete modules, namely the self-projective modules M which are perfect and, more generally, semiperfect in $\sigma[M]$. A module M is called Σ-*lifting* (Σ-*extending*) if the direct sum of any family of copies of M is lifting (extending, respectively).

The study of extending modules and lifting modules has its roots in the module theory of quasi-Frobenius rings. It is well-known that these rings are characterized as those over which every injective module is projective. Section 27 looks at the more general situation of modules M for which every injective module in $\sigma[M]$ is lifting. These are known as *Harada* modules and if $_RR$ is Harada then R is called a *left Harada ring*. A module M is called *co-Harada* if every projective module in $\sigma[M]$ is Σ-extending, and results due to Harada and Oshiro showing the interplay between Harada, co-Harada, and quasi-Frobenius rings are generalised here to a module setting. Finally, Section 28 carries this duality theme further by investigating modules M for which extending, or indeed all, modules in $\sigma[M]$ are lifting. We give extensive characterizations of both conditions. Specialising to the module ${}_RR$, we see that every extending R-module is lifting precisely when every self-projective R-module is extending and this characterizes artinian serial rings R. Similarly, every left R-module is extending if and only if all left R-modules are lifting, and this happens precisely when every left R-module is a direct sum of uniserial modules of length at most 2.

In the Appendix details are provided of two graph-theoretical techniques, namely Hall's Marriage Theorem and König's Graph Theorem. These are used in Chapter 3.

We have included a collection of exercises for most sections, in the hope that these will encourage the reader to further pursue the section material.

Notation

$\mathbb{Z}$ ($\mathbb{Q}$)	the integers (rationals)
card(I)	the cardinality of the set I
ACC (DCC)	the ascending (descending) chain condition
id, id_M, 1_M	identity map (of the set M)
$\mathrm{Ke}\, f$	the kernel of a linear map f
$\mathrm{Coke}\, f$	the cokernel of a linear map f
$\mathrm{Im}\, f$	the image of a map f
$g \circ f$, $f \diamond g$	composition of the maps $M \xrightarrow{f} N \xrightarrow{g} L$
M_K	short for $\bigoplus_{k \in K} M_k$ where the M_k are modules
Gen(M)	the class of M-generated modules, 1.1
Tr(M, N)	the trace of the module M in the module N, 1.1
Cog(M)	the class of M-cogenerated modules, 1.2
Re(N, M)	the reject of the module M in the module N, 1.2
$\sigma[M]$	the full subcategory of R-Mod subgenerated by M, 1.3
$K \trianglelefteq M$	K is a large (essential) submodule of M, 1.5
Soc(M)	the socle of the module M, 1.20
$K \ll M$	K is a small submodule of the module M, 2.2
$\nabla(M, N)$	$\{f \in \mathrm{Hom}(M, N) \mid \mathrm{Im}\, f \ll N\}$, 2.4
$\nabla(M)$	$\{f \in \mathrm{End}(M) \mid \mathrm{Im}\, f \ll N\}$, 2.4
Rad(M)	the radical of the module M, 2.8
Jac(R), $J(R)$	the Jacobson radical of the ring R, 2.11
$K \underset{M}{\overset{cs}{\hookrightarrow}} L$	$K \subset L$ is cosmall in M, 3.2
$N \overset{cc}{\hookrightarrow} M$	N is a coclosed submodule of M, 3.7
$K \overset{scc}{\hookrightarrow} M$	K is a strongly coclosed submodule of M, 3.12
u.dim(M)	the uniform dimension of the module M, 5.1
h.dim(M)	the hollow dimension of the module M, 5.2
$\mathcal{C}_M^{\trianglelefteq}$	the class of singular modules in $\sigma[M]$, 7.8
$\mathrm{Z}_M(N)$	the sum of the M-singular submodules of N, 7.8
$\mathrm{Re}_M^{\trianglelefteq}(N)$	the reject of the M-singular modules in N, 7.9
$\mathbb{S}[M]$, $\mathbb{S}$	the class of small modules in $\sigma[M]$, 8.2
$\mathrm{Tr}_{\mathbb{S}}(N)$	$\mathrm{Tr}(\mathbb{S}, N)$, the sum of the M-small submodules of N, 8.5
$\mathrm{Re}_{\mathbb{S}}(N)$	$\mathrm{Re}\,(N, \mathbb{S})$, reject of the M-small modules in N, 8.9
$K \underset{M}{\overset{cr}{\hookrightarrow}} L$	K is a corational submodule of L in M, 9.7

Contents

Chapter 1

Basic notions

1 Preliminaries

For basic definitions, theorems and notation, we refer the reader to texts in Module Theory, mainly to [363] and [85] and occasionally to [13], [100], and [189]. In this section, we recall some of the notions which will be of particular interest to us.

Throughout R will denote an associative ring with unit 1 and M will usually stand for a nonzero unital left R-module; if necessary we will write ${}_RM$ or M_R to indicate if we mean a left or right module over R. R-Mod and Mod-R denote the category of all unital left and right modules, respectively.

For R-modules M, N, we denote the group of R-module homomorphisms from M to N by $\text{Hom}_R(M, N)$, or simply $\text{Hom}(M, N)$, and the endomorphism ring of M by $\text{End}_R(M)$, or $\text{End}(M)$. The *kernel* of any $f \in \text{Hom}(M, N)$ is denoted by $\text{Ke}\, f$, and its *image* by $\text{Im}\, f$.

It is of some advantage to write homomorphisms of left modules on the right side of the (elements of the) module and homomorphisms of right modules on the left and we will usually do so. In this case we denote by $f \diamond g$ the composition of two left R-module morphisms $f : M \to N$, $g : N \to L$, and so $((m)f)g = (m)f \diamond g$, for any $m \in M$. If these are morphisms of right R-modules we write the morphisms on the left and their composition is (as usual) $g \circ f$, and $g(f(m)) = g \circ f(m)$, for any $m \in M$. When no confusion is likely we may take the liberty to delete the symbols $\circ$ or $\diamond$, respectively.

With this notation, a left R-module M is a right module over $S = \text{End}_R(M)$ and indeed an (R, S)-bimodule. (R, S)-submodules of M are referred to as *fully invariant* or *characteristic* submodules.

1.1. Trace. Let M be an R-module. Then an R-module N is called *M-generated* if it is a homomorphic image of a direct sum of copies of M. The class of all M-generated R-modules is denoted by $\text{Gen}(M)$.

The *trace of M in N* is the sum of all M-generated submodules of N; this is a fully invariant submodule of N which we denote by $\text{Tr}(M, N)$. In particular, $\text{Tr}(M, R)$, the trace of M in R, is a two-sided ideal.

Given a class $\mathbb{K}$ of R-modules, the *trace of $\mathbb{K}$ in N* is the sum of all K-generated submodules of N, for all $K \in \mathbb{K}$, and we denote it by $\text{Tr}(\mathbb{K}, N)$.

Given a left ideal I of R, we have $IM \in \text{Gen}(I)$, hence $IM \subset \text{Tr}(I, M)$. If I is idempotent, then $IM = \text{Tr}(I, M)$.

1.2. Reject. Dually, an R-module N is called *M-cogenerated* if it is embeddable in a direct product of copies of M. The class of all M-cogenerated modules is denoted by $\text{Cog}(M)$.

The *reject of M in N* is the intersection of all submodules $U \subset N$ for which N/U is M-cogenerated; this is a fully invariant submodule of N which we denote by $\text{Re}(N, M)$. For any class $\mathbb{K}$ of R-modules, the *reject of $\mathbb{K}$ in N* is the intersection of all submodules $U \subset N$ such that N/U is cogenerated by modules in the class $\mathbb{K}$, and we denote this by $\text{Re}(N, \mathbb{K})$. It follows that (see [363, 14.4])

$$\text{Re}(N, \mathbb{K}) = \bigcap \{\text{Ke}\, f \mid f \in \text{Hom}(N, K), K \in \mathbb{K}\}.$$

1.3. The category $\sigma[M]$. An R-module N is said to be *subgenerated by M* if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are all R-modules subgenerated by M (see [363, §15]). A module N is called a *subgenerator* in $\sigma[M]$ if $\sigma[M] = \sigma[N]$.

$\sigma[M]$ is closed under direct sums (coproducts), kernels and cokernels. The product of any family of modules $\{N_\lambda\}_\Lambda$ in the category $\sigma[M]$ is denoted by $\prod_\Lambda^M N_\lambda$. It is obtained as the trace of $\sigma[M]$ in the product in R-Mod, that is,

$$\textstyle\prod_\Lambda^M N_\lambda = \text{Tr}(\sigma[M], \prod_\Lambda N_\lambda).$$

1.4. Finitely presented and coherent modules in $\sigma[M]$. Let $N \in \sigma[M]$. Then N is said to be *finitely presented in $\sigma[M]$* if it is finitely generated and, in every exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ where L is finitely generated, K is also finitely generated (see [363, 25.1]).

N is called *locally coherent in $\sigma[M]$* if all its finitely generated submodules are finitely presented in $\sigma[M]$ (see [363, 26.1]).

N is called *locally noetherian* if all its finitely generated submodules are noetherian (see [363, § 27]).

While N is finitely generated in $\sigma[M]$ if and only if it is finitely generated in R-Mod, modules which are finitely presented in $\sigma[M]$ need not be finitely presented in R-Mod. For example, let M be a simple module which is not finitely R-presented; then M is finitely presented in $\sigma[M]$ but not in R-Mod.

In general there need not exist any nonzero finitely presented modules in $\sigma[M]$ (see [284, Example 1.7]). On the other hand, every finitely generated module is finitely presented in $\sigma[M]$ if and only if M is locally noetherian (see [363, 27.3]).

1.5. Large (essential) submodules. A submodule K of a nonzero module M is said to be *large* or *essential* (we write $K \trianglelefteq M$) if $K \cap L \neq 0$ for every nonzero submodule $L \subset M$. If all nonzero submodules of M are large in M, then M is called *uniform.* A monomorphism $f : N \to M$ is called *large* or *essential* if $\operatorname{Im} f \trianglelefteq M$. This can be characterised by the property that any $g : M \to L$ for which $f \diamond g$ is a monomorphism has to be a monomorphism.

1.6. Characterisation of uniform modules. *For M the following are equivalent:*

(a) *M is uniform;*

(b) *every nonzero submodule of M is indecomposable;*

(c) *for any module morphisms $K \xrightarrow{f} M \xrightarrow{g} N$, where $f \neq 0$, $f \diamond g$ injective implies that f and g are injective.*

Proof. (a)⇔(b). For any nonzero submodules $U, V \subset M$ with $U \cap V = 0$, the submodule $U + V = U \oplus V$ is decomposable.

(a)⇒(c). Let $f \diamond g$ be injective. Then clearly f is injective. Assume g is not injective. Then $\operatorname{Im} f \cap \operatorname{Ke} g \neq 0$ and this implies $\operatorname{Ke} f \diamond g \neq 0$, a contradiction.

(c)⇒(a). Let $K, V \subset M$ be submodules with $K \neq 0$ and $K \cap V = 0$. Then the composition of the canonical maps $K \to M \to M/V$ is injective and now (c) implies that $M \to M/V$ is injective, that is, $V = 0$. □

1.7. Singular modules. A module $N \in \sigma[M]$ is called *M-singular* or *singular in* $\sigma[M]$ provided $N \simeq L/K$ for some $L \in \sigma[M]$ and $K \trianglelefteq L$. The class of all M-singular modules is closed under submodules, factor modules and direct sums and we denote it by $\mathcal{C}_M^{\trianglelefteq}$. Notice that this notion is dependent on the choice of M (i.e., the category $\sigma[M]$) and obviously $\mathcal{C}_M^{\trianglelefteq} \subseteq \mathcal{C}_R^{\trianglelefteq}$.

1.8. M-injectivity. A module N is called *M-injective* if, for every submodule $K \subset M$, any morphism $f : K \to N$ can be extended to a morphism $M \to N$. In case M is M-injective, then M is also called *self-injective* or *quasi-injective.*

Any M-injective module is also L-injective for any $L \in \sigma[M]$. For $N \in \sigma[M]$, the *injective hull* $\widehat{N}$ in $\sigma[M]$ is an essential extension $N \trianglelefteq \widehat{N}$ where $\widehat{N}$ is M-injective. It is related to the injective hull $E(N)$ of N in R-Mod by the trace functor. More specifically, $\widehat{N} = \operatorname{Tr}(\sigma[M], E(N)) = \operatorname{Tr}(M, E(N))$ since $\operatorname{Tr}(\sigma[M], Q) = \operatorname{Tr}(M, Q)$ for any R-injective module Q (see, e.g., [363, §16, 17]).

1.9. Complement submodules. Given a submodule $K \subset M$, a submodule $L \subset M$ is called a *complement of K in M* if it is maximal in the set of all submodules $L' \subset M$ with $K \cap L' = 0$. By Zorn's Lemma, every submodule has a complement in M. A submodule $L \subset M$ is called a *complement submodule* provided it is the complement of some submodule of M. If L is a complement of K in M, then there is a complement $\overline{K}$ of L in M that contains K. By construction, $K \trianglelefteq \overline{K}$ and $\overline{K}$ has no proper essential extension in M; thus $\overline{K}$ is called an *essential closure* of K in M. In general essential closures need not be uniquely determined.

A submodule $K \subset M$ is called *(essentially) closed* if $K = \overline{K}$.

The following characterises closed submodules (see [363] and [85]).

1.10. Characterisation of closed submodules. *For a submodule $K \subset M$, the following are equivalent:*

(a) *K is a closed submodule;*

(b) *K is a complement submodule;*

(c) *$K = \widehat{K} \cap M$, where $\widehat{K}$ is a maximal essential extension of K in $\widehat{M}$.*

Notice that in (c), $\widehat{K}$ is a direct summand in the M-injective hull $\widehat{M}$.

1.11. Factors by closed submodules. *Let the submodule $K \subset M$ be a complement of $L \subset M$. Then:*

(1) *$(K+L)/K \trianglelefteq M/K$ and $K+L \trianglelefteq M$.*

(2) *If $U \trianglelefteq M$ and $K \subset U$, then $U/K \trianglelefteq M/K$.*

Proof. (1) is shown in [363, 17.6].

(2) First notice that $L \cap U \trianglelefteq L$ and hence K is also a complement of $L \cap U$. By (1), this implies that $(K + L \cap U)/K \trianglelefteq M/K$. Since $(K + L \cap U)/K \subset U/K$ we also get $U/K \trianglelefteq M/K$. □

1.12. Weakly M-injective modules. The module N is called *weakly M-injective* if every diagram in R-Mod

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M^{(\mathbb{N})} \\ & & \downarrow & & \\ & & N & & \end{array}$$

with exact row and K finitely generated, can be extended commutatively by a morphism $M^{(\mathbb{N})} \to N$, that is, the functor $\mathrm{Hom}(-, N)$ is exact with respect to the given row. Any weakly M-injective module $N \in \sigma[M]$ is M-generated and also $\widehat{M}$-generated (see [363, 16.11]). Notice that, putting $M = R$, weakly R-injective modules are just *FP-injective* modules (see [363, 35.8]).

1.13. π-injective and direct injective modules. A module M is said to be *π-injective* if, for any submodules $K, L \subset M$ with $K \cap L = 0$, the canonical monomorphism $M \to M/K \oplus M/L$ splits. Furthermore, M is called *direct injective* if for any direct summand $K \subset M$, any monomorphism $K \to M$ splits. Notice that any self-injective module M has both these properties (e.g., [85], [363]).

Regarding injective modules in $\sigma[M]$ and the locally noetherian property of M we have the following proved in [363, 27.3, 27.5].

1.14. Decomposition of injective modules. *For M the following are equivalent:*

(a) *M is locally noetherian;*

(b) *every weakly M-injective module is M-injective;*