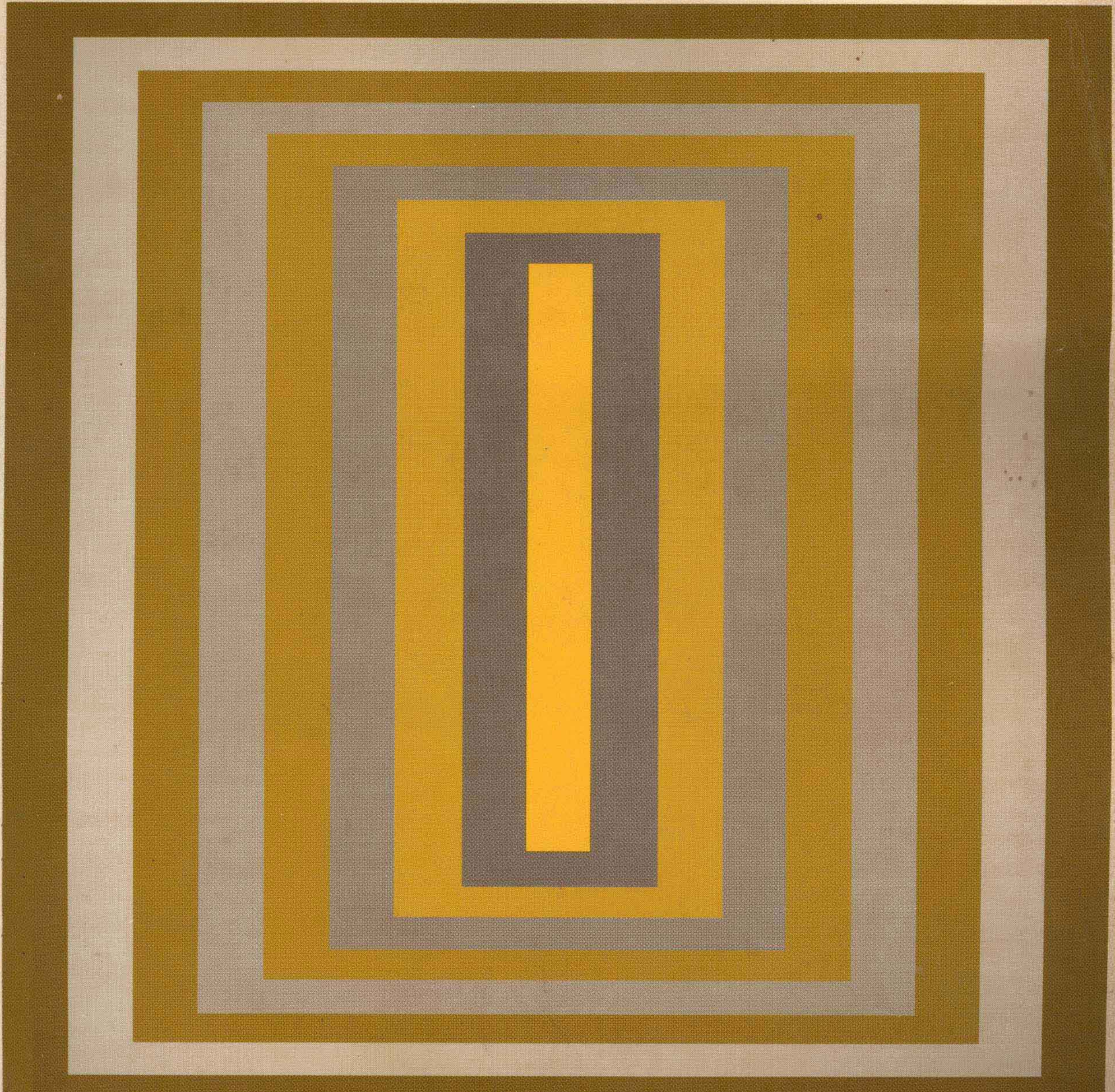
Bernard Kolman

Third Edition

ELEMENTARY LINEAR ALGEBRA



3rd Edition

ELEMENTARY LINEAR ALGEBRA

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List of Frequently Used Symbols

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P, Q	nonsingular matrices, pp. 67 (Exercise 17), 71
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$\{\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_n'\}$	the natural basis for R_n, pp. 102, 109
$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$,	the natural basis for R^3, p. 103
$\dim V$	the dimension of the vector space V, p. 107
$[\alpha]_s$	the coordinate vector of the vector α with respect to the ordered basis S for the vector space V, p. 115
$\lvert\alpha\rvert$	the length of the vector α, pp. 140, 158
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(α, β)	an inner product, pp. 143, 153
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$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$	an $n \times 1$ matrix whose entries are functions of t, p. 301
$\dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$	p. 301

Elementary Linear Algebra

To
Lillie,
Lisa,
and
Stephen

Preface

Over the past twelve years I have been pleased by the very favorable acceptance of the first and second editions of this book. Although many changes have been made in this edition, my basic objective has remained the same as in the first two editions: *to present the basic ideas of linear algebra in a manner that the student will find understandable.*

This textbook has been written for students who have completed a calculus course. It thus provides the student with his or her first experience in postulational or axiomatic mathematics while keeping in close touch with the computational aspects of the subject. Few subjects can claim to have such widespread applications in other areas of mathematics—multivariable calculus, differential equations, and probability theory, for example—as well as in physics, biology, chemistry, economics, psychology, sociology, and all fields of engineering. Engineering, the sciences, and the social sciences today are becoming more analytically oriented; that is, more mathematical in flavor, and the mere ability to manipulate matrices is no longer adequate. Linear algebra affords, at the sophomore level, an excellent opportunity to develop a capability for handling abstract concepts.

I have learned from experience that at the sophomore level, abstract ideas must be introduced quite gradually and must be based on some firm foundations. Thus we begin the study of linear algebra in **Chapter 1** with the treatment of matrices as mere arrays of numbers that arise naturally in the solution of systems of linear equations, a problem already familiar to the student. Methods for solving such systems of equations and matrix properties are considered in this chapter. In **Chapter 2,** we come to a more abstract notion, that of a vector space. We restrict our attention to finite-dimensional, real vector spaces. Here we tap some of the many geometric ideas that arise naturally. Thus we prove that an n-dimensional, real vector space is isomorphic to R^n, the vector space of all ordered n-tuples of real numbers, or the vector space of all $n \times 1$ matrices with real entries. Since R^n is but a slight generalization of R^2 and R^3, two- and three-dimensional space studied in the calculus, this shows that the notion of a finite-dimensional, real vector space is not as remote as it may have seemed when first introduced. **Chapter 3** covers inner product spaces and has a strong geometric orientation. **Chapter 4** deals with matrices and linear transformations; in it we consider the dimension theorems and also applications to the solution of systems of linear equations. **Chapter 5** introduces the basic properties of determinants and some of their applications. Chapter 6 considers eigenvalues and eigenvectors, and real quadratic forms. In this chapter we completely solve the diagonalization problem for symmetric matrices. **Chapter 7** provides an introduction to the important application of linear algebra in the solution of differential equations. It is possible to go from Section 6.1 directly to Chapter 7, providing an immediate application of the material in Section 6.1. Moreover, Section 6.3 provides an application of the material in Section 6.2. The **appendix** reviews some very basic material dealing with sets and functions. It can be consulted at any time, as needed.

In using the first two editions of this book, for a one-quarter, linear algebra course meeting four times a week, no difficulty has been encountered in covering the first six chapters (Section 6.3 was omitted and the optional material was not covered). A suggested pace for covering the basic material, based on 12 years of experience with the first two editions, follows.

Suggested Pace for Basic Material (Chapters 1 to 6, omitting all optional material and Section 6.3)

Chapter 1	8 lectures
Chapter 2	7 lectures
Chapter 3	3 lectures (For most students Section 3.1 is a review of known material.)
Chapter 4	7 lectures
Chapter 5	5 lectures
Chapter 6	5 lectures
	35 lectures

When teaching linear algebra, I have often found that the subject of eigenvalues and eigenvectors had to be covered much too hastily, because of time limitations, to do the material justice. This book has been written so that this very important topic can be comfortably covered in a one-quarter or one-semester course.

The exercises form an integral part of the text. Many of them are numerical in nature, whereas others are of a theoretical type. The answers to selected numerical exercises are provided in the answer section. An Answer Manual, containing answers to the remaining numerical exercises and solutions to all the theoretical exercises, is available (to instructors only) at no cost from the publisher.

I am pleased to express my thanks to the following reviewers: Edward Norman, Florida Technological University; the late Charles S. Duris, Drexel

University; William F. Trench, Drexel University; James O. Brooks, Villanova University; Stephen D. Kerr, Weber State College; and Herbert J. Nichol, Drexel University (emeritus). The numerous suggestions, comments, and criticisms of these people greatly improved the manuscript.

I thank James O. Brooks, Villanova University, for carefully preparing the Answer Manual, which contains answers to all the numerical exercises whose answers are not in the back of the book and solutions to all the theoretical exercises, and for critically reading the galley and page proofs. I thank Robert L. Higgins, Quantics and Drexel University, for helping to read galley and page proofs.

I also thank my wife, Lillie, and my daughter, Lisa, for their help with typing and other clerical tasks, and my son, Stephen, for help with the index, as well as all three for their support and encouragement.

I should also like to acknowledge the students at Drexel University who used the first two editions and the instructors from a number of institutions who communicated to me their classroom experiences with the first two editions.

Finally, I should like to thank Wayne Yuhasz, Senior Editor; Elaine W. Wetterau, Production Supervisor; and the entire staff of the Macmillan Publishing Co., Inc., for their interest and unfailing cooperation in all phases of this project. To all these goes a sincere expression of thanks.

B. K.

New Features in the Third Edition

- Section 2.2 of the second edition, "Linear Independence and Basis," has been divided into two sections to facilitate their study.
- Chapter 0 of the second edition, "Preliminaries," has been made an appendix.
- More exercises at all levels.
- More illustrative examples.
- Biographical sketches of leading contributors to the subject.
- Improved clarity of writing in many places.
- New, more open, modern design for improved legibility and appearance.

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Contents

1 Linear Equations and Matrices

1.1. Systems of Linear Equations

One of the most frequently recurring practical problems in almost all fields of study—such as mathematics, physics, biology, chemistry, economics, all phases of engineering, operations research, the social sciences, and so forth—is that of solving a system of linear equations. The equation

$$b = a_1x_1 + a_2x_2 + \cdots + a_nx_n, \tag{1}$$

which expresses b in terms of the variables $x_1, x_2, \ldots, x_n$ and the constants $a_1, a_2, \ldots, a_n$, is called a **linear equation.** In many applications we are given b and must find numbers $x_1, x_2, \ldots, x_n$ satisfying (1).

A **solution** to a linear equation (1) is a sequence of n numbers $s_1, s_2, \ldots, s_n$ such that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted in (1). Thus $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

Note: The appendix, which was Chapter 0 in the first two editions, reviews some very basic material dealing with sets and functions. It can be consulted at any time, as needed.

More generally, a **system of m linear equations in n unknowns,** or a **linear system,** is a set of m linear equations each in n unknowns. A linear system can be conveniently written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{2}$$

Thus the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

In (2) the a_{ij} are known constants. Given values of $b_1, b_2, \ldots, b_n$, we want to find values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A **solution** to a linear system (2) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which have the property that each equation in (2) is satisfied when $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ are substituted.

If the linear system (2) has no solution, it is said to be **inconsistent;** if it has a solution, it is called **consistent.** If $b_1 = b_2 = \cdots = b_m = 0$, then (2) is called a **homogeneous system.** The solution $x_1 = x_2 = \cdots = x_n = 0$ to a homogeneous system is called the **trivial solution.** A solution to a homogeneous system in which not all of $x_1, x_2, \ldots, x_n$ are zero is called a **nontrivial solution.**

Consider another system of r linear equations in n unknowns:

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ c_{r1}x_1 + c_{r2}x_2 + \cdots + c_{rn}x_n &= d_r. \end{aligned} \tag{3}$$

We say that (2) and (3) are **equivalent** if they both have exactly the same solutions.

Example 1. The linear system

$$\begin{aligned} x_1 - 3x_2 &= -7 \\ 2x_1 + x_2 &= 7 \end{aligned} \tag{4}$$