INTRODUCTION TO LINEAR ALGEBRA

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PREFACE

Linear algebra is an important component of undergraduate mathematics, particularly for students majoring in the scientific, engineering, and social science disciplines. At the practical level, matrix theory and the related vector space concepts provide a language and a powerful computational framework for posing and solving important problems. Beyond this, elementary linear algebra is a valuable introduction to mathematical abstraction and logical reasoning because the theoretical development is self-contained, consistent, and accessible to most students.

Therefore this book stresses both practical computation and theoretical principles and centers around

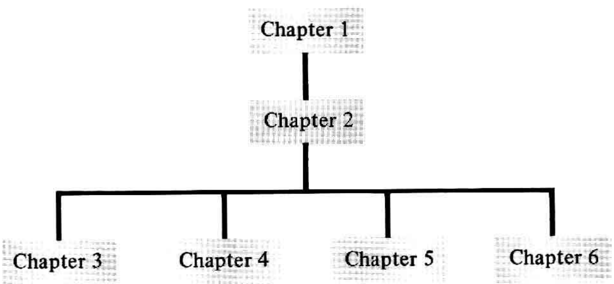
matrix theory and systems of linear equations,

elementary vector-space concepts, and

the eigenvalue problem.

The text is designed for a one-term course at the late-freshman or sophomore level and is organized so that these three topics can be covered in even a short (10-week) course. However there is enough material for a more leisurely paced course or for a sophomore/junior course at the level of the usual "advanced engineering mathematics" sequence.

To provide a measure of flexibility, we have written Chapters 3, 4, 5, and 6 so that they are essentially independent and can be taken in any order once Chapters 1 and 2 are covered. In particular Chapter 5 (Determinants) can be covered before Chapter 3(Eigenvalues) if desired, all or portions of Chapter 4 (Abstract Vector Spaces) can be covered at any time, etc. The chapter dependencies are given schematically by the following diagram.



For example, a short course at the beginning level can be built around Sections 1.1–1.8, 2.1–2.4, 3.1–3.6 and can be supplemented by other topics such as least-squares approximations, abstract vector spaces, determinant theory, etc. (Also a brief review of vector geometry is given in the appendix.)

We have introduced the idea of linear independence very early (in Section 1.6) and use it extensively thereafter. Linear independence is one unifying thread running throughout the text. In addition, the early introduction of linear independence will ease the transition from the concrete material in Chapter 1 to the more abstract ideas in Chapter 2, such as subspace, basis, and dimension in R^n. The treatment of the eigenvalue problem in Chapter 3 is more detailed than that in most books at this level, and we emphasize computation and application of eigenvalues as well as the theoretical foundations. In Sections 3.2 and 3.3, eigenvalues are introduced in the traditional way, including a brief discussion of determinants. In Sections 3.7–3.9 we use similarity transformations to explain reduction to Hessenberg form, and then we show the parallels between Hessenberg form for the eigenvalue problem and echelon form for the problem of solving $A\mathbf{x} = \mathbf{b}$. The material in Sections 3.7–3.9 is theoretically complete, computationally relevant, and developmentally important for ideas such as the Cayley–Hamilton theorem and generalized eigenvectors. Finally if a more comprehensive treatment of determinants is desired before introducing eigenvalues, Chapter 5 can be covered before Chapter 3.

Our applications have been drawn from differential equations and difference equations and from problems of interpolating data and finding best least-squares fits to data. In particular in their major curriculum most students have been exposed to problems of drawing curves that fit experimental or empirical data, and they can appreciate the techniques from linear algebra that can be applied to these problems. In Chapter 6 we also discuss numerical methods for linear algebra and include a number of simple computer programs that can be used to implement the numerical methods. These programs are in the form of well-documented subroutines and include, for example, programs that solve $(n \times n)$ systems $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ may contain complex entries, programs that use Householder transformations to reduce A to Hessenberg form, and programs that use the Givens algorithm to find the eigenvalues and eigenvectors for a symmetric matrix.

As far as the nature of the material permits, we have tried to organize each section so that it can be covered in one period; where such organization is not possible, we have tried to follow with a short section. The individual sections contain a number of examples, many worked in extreme detail so that the student can read them unassisted. Some sections also contain optional material, marked with a dagger (†); this material can be omitted without any loss of continuity. In particular, entire sections, such as †2.5 and †2.6, may be skipped. In keeping with the introductory nature of the text, many of the exercises are routinely computational while many of the theoretical exercises contain fairly strong hints and should be relatively easy. Some of the exercises are designed also to motivate and give concrete illustrations of topics in later sections. We have made a determined effort to construct the exercises so that the numerical calculations proceed smoothly and do not obscure the point of the problem, and we have tried also to be certain that the answer key is correct.

We have attempted to make the text theoretically complete and self-contained although the inherent nature of the material means that some of the topics demand more mathematical maturity. In particular, Section †3.10 and some of the later sections in Chapter 4 are probably not appropriate for most beginning students.

Finally we would like to express our appreciation and gratitude to the following reviewers for their assistance during the development of the manuscript: Betty J. Barr from the University of Houston, Monte Boisen from Virginia Polytechnic Institute and State University, William A. Brown from the University of Southern Maine, Alexander Hahn from the University of Notre Dame, Steven K. Ingram from Norwich University, Robert M. McConnel from The University of Tennessee, and Gerald J. Roskes from Queens College of The City University of New York.

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CONTENTS

1

MATRICES AND SYSTEMS OF LINEAR EQUATIONS

1.1 INTRODUCTION AND GAUSS ELIMINATION

In science, engineering, and the social sciences, one of the most important and frequently occurring mathematical problems is finding a simultaneous solution to a set of linear equations involving several unknowns. A simple example is the problem of finding values for x_1 and x_2 that simultaneously satisfy the equations

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1. \end{aligned}$$

This system of equations is easily solved by observing that the first equation requires $x_1 = 3 - x_2$. Inserting that relationship in the second equation gives $(3 - x_2) - x_2 = 1$, which yields $x_2 = 1$; and hence $x_1 = 2$. An alternative approach is to add the two equations together finding $(x_1 + x_2) + (x_1 - x_2) = 4$, or $2x_1 = 4$. Thus again $x_1 = 2$ and $x_2 = 1$ is the solution. This simple system of two equations in two unknowns is quite easy to solve, but obviously some sort of systematic procedure is needed to handle the more complicated problems that arise in practice. To answer this need, the first topic of this chapter is the general problem of solving a system of m linear equations in n unknowns; and the goal is to develop a *systematic* and computationally efficient method for solving such systems. Whenever it is appropriate, we will also comment on the practical aspects of solving systems of linear equations on a digital computer.

An $(m \times n)$ ***system of linear equations*** is the system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned} \tag{1.1}$$

In (1.1) the coefficients a_{ij} for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ are known constants as are b_1, $b_2, \ldots, b_m$. The unknowns in the system are designated $x_1, x_2, \ldots, x_n$; and a

solution to (1.1) is a set of n numbers $x_1, x_2, \ldots, x_n$ that satisfies each of the m equations in (1.1)*. The subscript notation in (1.1) is necessary to identify the variables and the constants. For example in the general (3×3) system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned}$$

the subscript notation signifies that a_{32} is the coefficient of the second variable, x_2, in the third equation. The system (1.1) is called linear since each equation is of the first degree in each of the variables $x_1, x_2, \ldots, x_n$. That is, none of the equations contains terms such as x_1^2, x_1x_2, $\sin x_3$, etc. An example of a system of *nonlinear* equations is

$$\begin{aligned} x_1^2 + x_2^2 &= 9 \\ x_1 + x_2 &= 1. \end{aligned}$$

(The solutions of this system are the points on the intersection of the circle $x_1^2 + x_2^2 = 9$ and the line $x_1 + x_2 = 1$.)

The linear system (1.1) is called ***consistent*** if it has at least one solution and is called ***inconsistent*** if there is no solution. In (1.1) we have placed no restriction on the relative sizes of m and n; so we may have more equations than unknowns $(m > n)$, more unknowns than equations $(m < n)$, or an equal number of equations and unknowns. Before discussing methods for solving (1.1), we show by example (even when $m = n$) that a linear system may have no solution, a unique solution, or infinitely many solutions. As one simple example, the (2×2) system

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 4 \end{aligned}$$

is consistent but has infinitely many solutions. Clearly any values of x_1 and x_2 that are related by $x_2 = 2 - x_1$ (for example, $x_1 = 1$, $x_2 = 1$ or $x_1 = -3$, $x_2 = 5$) will satisfy both equations. On the other hand the (2×2) system

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}$$

is clearly inconsistent. Finally the first example given,

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1, \end{aligned}$$

is a consistent system with the unique solution $x_1 = 2$, $x_2 = 1$.

The examples above are typical because a consistent system of linear equations

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* For clarity of presentation, we will assume throughout this chapter that the constants a_{ij} and b_i are real numbers although all statements are equally valid for complex constants. When we consider eigenvalue problems, we will occasionally encounter linear systems having complex coefficients; but the solution technique is no different.

always has either exactly one solution or infinitely many solutions. Each of these three systems has a simple geometric interpretation, as shown in Fig. 1.1. Geometrically a solution to a (2×2) system of linear equations represents a point of intersection of two lines; and there are three possibilities: the two lines may be coincident (the same line), they may be parallel (having no points of intersection), or the lines may intersect in exactly one point.

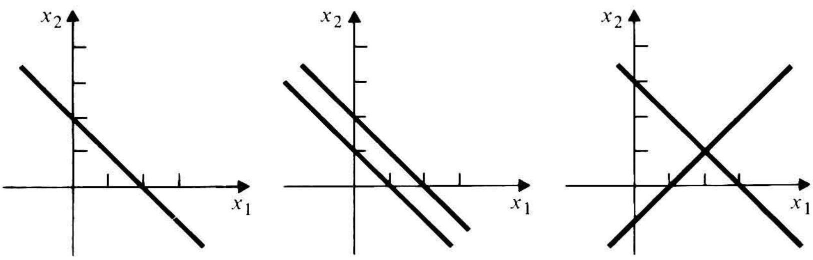

Fig. 1.1 Geometry of (2×2) linear systems.

The simplest and best-known method for solving a general $(m \times n)$ system as in (1.1) is the familiar variable-elimination technique known as Gauss elimination. This technique is computationally practical and is the procedure most widely used in computer-software packages that are designed to solve systems of linear equations. We shall illustrate Gauss elimination with an example, and then follow with a more careful description of this solution process. Consider the (3×3) linear system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 + x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 - x_3 &= 1. \end{aligned}$$

If we multiply the first equation by -1 and add the result to the second equation, and then multiply the first equation by -2 and add the result to the third equation, we obtain the following system where x_1 has been eliminated from the second and third equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ -x_2 + 2x_3 &= -4 \\ -5x_2 - 3x_3 &= -7. \end{aligned}$$

For the moment we ignore the first equation and eliminate x_2 by multiplying the second equation by -5 and adding the result to the third equation. In this way we eliminate x_2 from the third equation and obtain a system having a "triangular form":

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ -x_2 + 2x_3 &= -4 \\ -13x_3 &= 13. \end{aligned}$$

From the third equation we find $x_3 = -1$. Using $x_3 = -1$ in the second equation yields $-x_2 - 2 = -4$, or $x_2 = 2$. Finally in the first equation, $x_1 + 4 - 1 = 4$, or $x_1 = 1$. We can easily verify that $x_1 = 1, x_2 = 2, x_3 = -1$ is a solution of the original system. (Geometrically the solution $x_1 = 1$, $x_2 = 2$, $x_3 = -1$ is the point of intersection of the three planes: $x_1 + 2x_2 + x_3 = 4$, $x_1 + x_2 + 3x_3 = 0$, $2x_1 - x_2 - x_3 = 1$.)

In general the steps of Gauss elimination follow almost exactly as in the example above. The objective of Gauss elimination is to transform the system (1.1)

$$\begin{array}{ccccccc} a_{11}x_1 & + a_{12}x_2 & + & \dots & + a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + a_{22}x_2 & + & \dots & + a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + & \dots & + a_{mn}x_n & = & b_m \end{array}$$

systematically into an equivalent linear system that is easier to solve. By a system that is ***equivalent*** to (1.1), we mean a system with unknowns $x_1, x_2, \dots, x_n$ that has exactly the same solution set as (1.1). To be more precise, consider the $(r \times n)$ system:

$$\begin{array}{ccccccc} c_{11}x_1 & + c_{12}x_2 & + & \dots & + c_{1n}x_n & = & d_1 \\ c_{21}x_1 & + c_{22}x_2 & + & \dots & + c_{2n}x_n & = & d_2 \\ \vdots & & & & \vdots & & \vdots \\ c_{r1}x_1 & + c_{r2}x_2 & + & \dots & + c_{rn}x_n & = & d_r. \end{array} \tag{1.2}$$

Then (1.1) and (1.2) are equivalent systems if any solution of (1.1) is a solution of (1.2) and any solution of (1.2) is a solution of (1.1).

Gauss elimination [that is, the transformation of (1.1) into an easily solved but equivalent system] is carried out using the three ***elementary operations*** listed below:

1. interchange of two equations,
2. multiplication of an equation by a nonzero scalar, and
3. addition of a constant multiple of one equation to another.

In Exercise 11, Section 1.1, the reader is asked to prove that the application of any of these elementary operations to a system will produce an equivalent system. A precise description of Gauss elimination is most easily given in the language of matrix theory; so we defer this precise description until Section 1.2. However in order at least to describe the essence of Gauss elimination and to allow the reader some practice in solving linear systems, we will indicate briefly how the basic steps are carried out.

The first stage of Gauss elimination as applied in solving (1.1) is to use the first equation and the three elementary operations to eliminate x_1 from equations 2, 3, ..., m. This elimination is easily done (if $a_{11} \neq 0$) by multiplying the first equation by $-a_{i1}/a_{11}$ and adding the result to the ith equation, for $i = 2, 3, \dots, m$.

For each i, $i \geqslant 2$, the effect is that of replacing the original ith equation by a new equation in which x_1 does not appear, and of producing a system equivalent to (1.1):

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\ \vdots \qquad \qquad \vdots \quad &\quad \vdots \\ a'_{m2}x_2 + \ldots + a'_{mn}x_n &= b'_m. \end{aligned} \tag{1.3}$$

In (1.3) the primed terms indicate the equations that result when multiples of the first equation are added to equations 2, 3, . . . , m of (1.1). In particular the primed terms in (1.3) are

$$a'_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j} \qquad \text{and} \qquad b'_i = b_i - \frac{a_{i1}}{a_{11}}b_1.$$

If $a_{11} = 0$ in (1.1), then we would have to interchange equations in (1.1) in order to eliminate x_1. We will consider complications of this sort more thoroughly in the next section. Also note that if we can solve the last $m-1$ equations of (1.3) for $x_2, x_3, \ldots, x_n$, then we can determine x_1 from the first equation of (1.3). Thus in (1.3) we have essentially reduced the problem of solving (1.1) to the smaller problem of solving a system of $m-1$ equations in $n-1$ unknowns.

The next step in Gauss elimination is to use the second equation of (1.3) to eliminate x_2 from equations 3, 4, . . . , m. If $a'_{22} \neq 0$, this elimination is accomplished by multiplying the second equation of (1.3) by $-a'_{i2}/a'_{22}$ and adding the result to the ith equation of (1.3), for $i = 3, 4, \ldots, m$. Carrying this out, we obtain

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\ \vdots \qquad \qquad \vdots \quad &\quad \vdots \\ a''_{m3}x_3 + \ldots + a''_{mn}x_n &= b''_m. \end{aligned} \tag{1.4}$$

Moreover (1.4) is equivalent to (1.3), and (1.3) is equivalent to (1.1) since they were derived by elementary operations. Thus any solution of (1.4) is a solution of (1.1), and any solution of (1.1) is a solution of (1.4).

Although Gauss elimination may proceed not quite so neatly as we are describing it, our objective is to produce a linear system that is equivalent to (1.1) and that has the following form (in the case $m \leqslant n$):

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1m}x_m + \ldots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \ldots + c_{2m}x_m + \ldots + c_{2n}x_n &= d_2 \\ \vdots \qquad \qquad \vdots \quad &\quad \vdots \\ c_{mm}x_m + \ldots + c_{mn}x_n &= d_m. \end{aligned} \tag{1.5}$$

If the coefficients $c_{11}, c_{22}, \ldots, c_{mm}$ are all nonzero, then we can solve (1.5) relatively easily; in the special case that $m = n$, the solution will also be unique. The solution process for (1.5) is called ***backsolving*** and proceeds in the obvious fashion. That is, if we allow $x_{m+1}, \ldots, x_n$ to be variables (or free parameters), then x_m is determined from the last equation of (1.5) by

$$x_m = (d_m - c_{m,m+1}x_{m+1} - \cdots - c_{mn}x_n)/c_{mm}.$$

This formula for x_m is then inserted in the $(m-1)$st equation of (1.5) to determine x_{m-1}. In general, x_i is determined from the ith equation of (1.5) using $x_{i+1}, \ldots, x_n$. We will treat the case that some $c_{ii} = 0$ and the case that $m > n$ in the next section, using the idea of an echelon matrix. (Echelon matrices will serve also as convenient notational devices allowing us to organize the steps of Gauss elimination in a compact fashion.) We conclude this section with a number of simple examples that illustrate Gauss elimination.

EXAMPLE 1.1 To demonstrate Gauss elimination, we use the procedure to solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 1 \\ -3x_1 + x_2 - 2x_3 &= -5. \end{aligned} \tag{1.6}$$

Following the description given above, we multiply the first equation by -2 and add the result to the second, and then multiply the first equation by 3 and add the result to the third (in the previous notation, $-a_{21}/a_{11} = -2$ and $-a_{31}/a_{11} = 3$). These steps yield an equivalent system of the form (1.3):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 5x_2 - 3x_3 &= -3 \\ -5x_2 + x_3 &= 1. \end{aligned} \tag{1.6a}$$

The final step for this simple example is to eliminate x_2 from the third equation by adding the second equation to the third (in the notation used above, $-a'_{32}/a'_{22} = 1$). We thus arrive at a "triangular" system [of the form (1.5)], which is equivalent to (1.6):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 5x_2 - 3x_3 &= -3 \\ -2x_3 &= -2. \end{aligned} \tag{1.6b}$$

If we backsolve, the last equation of this triangular system yields $x_3 = 1$; and by having $x_3 = 1$, the second equation requires $5x_2 = 0$, or $x_2 = 0$. Given that $x_3 = 1$ and $x_2 = 0$, then the first equation yields $x_1 = 1$. Clearly the only solution of the triangular system (1.6b) is $x_1 = 1$, $x_2 = 0$, $x_3 = 1$; and since the triangular system and the original system (1.6) are equivalent, we know that the only solution of (1.6) is also given by $x_1 = 1$, $x_2 = 0$, $x_3 = 1$.

EXAMPLE 1.2 In this example we use Gauss elimination to show that the system (1.7) has no solution:

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ -3x_1 + x_2 - 2x_3 &= 1.1. \end{aligned} \tag{1.7}$$

Proceeding as in Example 1.1, we multiply the first equation by -2 and add the result to the second equation, and then multiply the first equation by 3 and add to the third equation. These steps give

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 5x_2 + 5x_3 &= 5 \\ -5x_2 - 5x_3 &= -4.9. \end{aligned}$$

Next adding the second equation to the third, we obtain a system *equivalent* to (1.7):

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 5x_2 + 5x_3 &= 5 \\ 0x_3 &= .1. \end{aligned} \tag{1.7a}$$

Because there is no number x_3 such that $0x_3 = .1$, the system (1.7a) is an inconsistent system. Since (1.7a) and (1.7) are equivalent systems, then (1.7) has no solution. In a later section we will develop a procedure for finding a "best" approximate solution to (1.7), that is, a procedure for finding a set of values x_1, x_2, x_3 that comes closest to solving (or "best fits") the three equations in (1.7).

EXAMPLE 1.3 This example shows how Gauss elimination can be used to exhibit all the solutions of a consistent system that has infinitely many solutions. Applying Gauss elimination to

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ -3x_1 + x_2 - 2x_3 &= 1, \end{aligned} \tag{1.8}$$

we find after eliminating x_1 that (1.8) is equivalent to

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 5x_2 + 5x_3 &= 5 \\ -5x_2 - 5x_3 &= -5. \end{aligned}$$

Multiplying the second equation by 1 and adding the result to the third, we next obtain

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 5x_2 + 5x_3 &= 5 \\ 0x_3 &= 0. \end{aligned} \tag{1.8a}$$

This case is unlike (1.7a) because the last equation $0x_3 = 0$ can be satisfied by any value of x_3. In fact we may as well dispense completely with the last equation of (1.8a); then we see that (1.8) is equivalent to the (2×3) system

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 5x_2 + 5x_3 &= 5. \end{aligned} \tag{1.8b}$$

Backsolving (1.8b), we find $x_2 = 1 - x_3$ from the second equation of (1.8b); and then $x_1 = -2 + x_3 + 2x_2 = -2 + x_3 + 2(1 - x_3) = -x_3$ from the first equation. Thus the solution of (1.8) is

$$\begin{aligned} x_1 &= \quad -x_3 \\ x_2 &= 1 - x_3 \end{aligned}$$

where x_3 is a free parameter. For example $x_3 = 0$, $x_2 = 1$, $x_1 = 0$ is a solution as is $x_3 = 4$, $x_2 = -3$, $x_1 = -4$.

EXAMPLE 1.4 In this example we solve a simple "rectangular" system, the (3×4) system

$$\begin{aligned} x_2 + x_3 - x_4 &= 0 \\ x_1 - x_2 + 3x_3 - x_4 &= -2 \\ x_1 + x_2 + x_3 + x_4 &= 2. \end{aligned}$$

We immediately observe that the first equation cannot be used to eliminate x_1 from the second and third equations. This problem is easily overcome by interchanging the first and second equations, and we obtain an equivalent system

$$\begin{aligned} x_1 - x_2 + 3x_3 - x_4 &= -2 \\ x_2 + x_3 - x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 2. \end{aligned}$$

Eliminating x_1, we have

$$\begin{aligned} x_1 - x_2 + 3x_3 - x_4 &= -2 \\ x_2 + x_3 - x_4 &= 0 \\ 2x_2 - 2x_3 + 2x_4 &= 4. \end{aligned}$$

Finally we eliminate x_2 from the third equation by multiplying the second equation by -2 and adding the result to the third:

$$\begin{aligned} x_1 - x_2 + 3x_3 - x_4 &= -2 \\ x_2 + x_3 - x_4 &= 0 \\ -4x_3 + 4x_4 &= 4. \end{aligned}$$

We can solve the last equation for x_3 in terms of x_4 (and find $x_3 = x_4 - 1$) or for x_4 in terms of x_3 (and find $x_4 = 1 + x_3$). Selecting $x_3 = x_4 - 1$, we have $x_2 = 1$ from the second equation and $x_1 = 2 - 2x_4$. If we had chosen to write the solution in terms of x_3 instead of x_4, we would have used $x_4 = 1 + x_3$, then found $x_2 = 1$ and finally $x_1 = -2x_3$. Thus for this example we can give two forms for the solution set:

$$\begin{aligned} x_1 &= 2 - 2x_4 \\ x_2 &= 1 \\ x_3 &= x_4 - 1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= \quad -2x_3 \\ x_2 &= 1 \\ x_4 &= 1 + x_3. \end{aligned}$$

We could have rearranged variables in the original system so that Gauss elimination resulted in solving for x_3 and x_4 in terms of x_1. For this particular example we will always obtain $x_2 = 1$, and hence it would not have been possible to solve for x_1, x_3, and x_4 in terms of x_2. What is invariant about these solution sets is that $x_2 = 1$ and that any two of the remaining variables may be expressed in terms of the third. Hence we have one independent (unconstrained) variable and three dependent (con-