CONTEMPORARY MATHEMATICS

269

# Laminations and Foliations in Dynamics, Geometry and Topology

Proceedings of the Conference on Laminations and Foliations in Dynamics, Geometry and Topology
May 18–24, 1998
SUNY at Stony Brook

Mikhail Lyubich
John W. Milnor
Yair N. Minsky
Editors



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# Laminations and Foliations in Dynamics, Geometry and Topology

# Introduction

The concepts of lamination and foliation have recently gained fresh strength. They have enabled advances in long-standing problems in hyperbolic geometry, led to new concepts in topology, witnessed solution of classical problems on analytic differential equations, contributed to the renormalization theory of one-dimensional dynamical systems, and played a crucial role in the foundations of higher dimensional holomorphic dynamics.

In spite of their deep relations, these different topics have developed as distinct research fields, with little interaction between their practitioners. In an effort to overcome this separation and bring these diverse points of view closer together, we held a conference at Stony Brook in May of 1998. The conference brought together experts in the different aspects of lamination and foliation theory, as well as many young researchers and graduate students.

## The Present Volume

All but the last of the papers in this volume are based on lectures given at the Stony Brook conference. The first two are based on minicourses from the conference:

- **Geodesic laminations,** by Francis Bonahon.
  One-dimensional geodesic laminations on hyperbolic surfaces arose in Thurston's work on Teichmüller spaces and hyperbolic 3-manifolds, where they play a central role. For example, in Teichmüller theory they allow one to describe asymptotic properties of degenerating hyperbolic metrics on the surface. In hyperbolic 3-manifolds they describe the bending of convex hulls and the asymptotic geometry of ends. This presentation sets out the basic theory of geodesic laminations, discussing both its topological aspects and the more analytical ones arising from consideration of transverse measures and distributions. Some applications are given to the geometry of convex-hull boundaries in hyperbolic 3-manifolds.
- **Laminations, Foliations and the Topology of 3-Manifolds,** by David Gabai.
  The notion of (two-dimensional) taut foliation in a 3-manifold, and the related notion of essential lamination, generalize in some ways the properties of incompressible surfaces and have been much developed and applied by Gabai, Hatcher, Oertel, Thurston and others. A recurring theme in this minicourse is the notion of "minimal position" of curves or surfaces with respect to a taut foliation or essential lamination, and the applications of these ideas to answering deep questions about knot theory, covering spaces and algebraic topology of 3-manifolds.

- **Dicritical singularities of holomorphic vector fields,** by Cesar Camacho.
  This paper discusses the local structure of singularities of holomorphic vector fields in $\mathbb{C}^2$ and of the associated foliations. It describes the resolution procedure and the monodromy group, and then gives criteria for existence of meromorphic and Liouvillian integrals, and introduces two new invariants of the singularity.
- **Dynamics of $\mathbb{P}^2$: examples,** by John Eric Fornaess and Nessim Sibony.

  The study of holomorphic dynamics in two or more complex dimensions is a relatively new but rapidly developing field. This paper explores the dynamics of holomorphic maps from the complex projective plane to itself, providing a number of surprising and instructive examples.

- **Rational laminations of complex polynomials,** by Jan Kiwi.
  If $f$ is a monic polynomial map with connected Julia set, then the associated *rational lamination* is an equivalence relation on $\mathbb{Q}/\mathbb{Z}$, where two rational angles are equivalent if and only if the associated external rays land at a common point of the Julia set. This concept was perhaps first introduced by McMullen, although it is based on earlier work by Thurston, Douady and Hubbard. Two polynomials with the same rational lamination are sometimes said to be *combinatorially equivalent.* The present paper provides a big step towards understanding polynomials of higher degree by giving a complete characterization of those equivalence relations which can arise as rational laminations. The results are easy to state, but quite difficult to prove.

- **Actions of discrete groups on complex projective spaces,** by José Seade and Alberto Verjovšky.
  By definition, a higher dimensional "*Complex Kleinian Group*" is a discrete subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$ which act on the projective space $P^n_{\mathbb{C}}$, $n > 1$ in such a way that the domain of discontinuity is non-empty. This paper provides a survey of the field. In particular, it discusses higher dimensional analogues of Fuchsian groups. Using twistor theory, it shows that every Kleinian group of conformal automorphisms of $S^4$ gives rise to a complex Kleinian group of automorphisms of $P^3_{\mathbb{C}}$.

- **Dynamics of singular holomorphic foliations on the complex projective plane,** by Saeed Zakeri.
  Any holomorphic foliation of the complex projective plane by curves, with only mild singularities, is induced by a polynomial vector field on the plane. This is an outline of the theory, including discussion of concepts of 'degree', of monodromy, density of leaves, and ergodicity, as well as the possible existence of foliations with a non-trivial minimal set.

## The Stony Brook Conference

A number of the lectures at the conference are not represented in this volume. First there were two further minicourses:

- **Dynamics and currents in $\mathbb{C}^2$,** by E. Bedford and J. Smillie.
  Classically, laminations arise in the theory of hyperbolic dynamical systems as families of stable and unstable manifolds. There are natural "Ruelle-Sullivan currents" associated with these laminations. Remarkably, in two-dimensional holomorphic dynamics (iteration theory of polynomial diffeomorphisms of $\mathbb{C}^2$), currents come first as distributional $\partial\bar{\partial}$-derivatives of certain pluripotential Green's functions. These can then be interpreted as geometric currents supported on (1-complex-dimensional) laminations, which provide deep insight into the topological dynamics of the map.
  [For much of this material, see for example the series of papers "*Polynomial diffeomorphisms of* $\mathbb{C}^2$" by Bedford and Smillie, in Invent. Math. **103** (1991) and **112** (1993) (joint with Lyubich), J. Amer. Math. Soc. **4** (1991), Math. Ann. **294** (1992), J. Geom. Anal. **8** (1998), Ann. Math. **148** (1998), and Ann. Sci. Éc. Norm. Sup. **32** (1999).]

- **Riemann surface laminations: uniformization and meromorphic functions,** by É. Ghys.
  This minicourse focused on two foundational problems in the theory of Riemann surface laminations: the problem of embedding a Riemann surface lamination into a projective space, and the uniformization problem for Riemann surface laminations. These are motivated by the classical theory of Riemann surfaces, the Poincaré-Bendixon theory, and a conjecture of Camacho. The lectures developed a conceptual background for the discussion (foliated cycles, harmonic currents, divisors...) and presented many fascinating examples (e.g., an example of a non-flat lamination with all leaves parabolic) and several nice results (e.g., some necessary conditions for the embedding into a projective space) in this direction.
  [An exposition of these ideas can be found in "*Laminations par surfaces de Riemann*", Panoramas et Synthèses **8** (1999), 49–95.]

In addition to the minicourses, there were a number of one-hour talks on current research topics:

- C. Camacho. *Complex foliations near a singularity.*
- A. Connes. *The Riemann flow and the Zeros of Zeta.*
- Ya. Eliashberg. *Foliations and contact structures on 3-manifolds.*
- S. Fenley. *Foliations with good geometry.*
- J.-E. Fornaess. *Remarks on dynamics on* $\mathbb{P}^2$.
- X. Gomez-Mont. *The attractor of a Riccati equation and the foliated geodesic flow.*
- A. Hatcher. *Kontsevich's conjecture on diffeomorphism groups of 3-manifolds.*
- J. H. Hubbard. *Foliations of domains in* $\mathbb{C}^2$ *with pluri-harmonic functions.*
- S. Hurder. *The Global Geometry of Riemannian Foliations.*
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- J. Kiwi. *Rational laminations of complex polynomials.*

– L. Mosher. *Laminations and solvable groups.*
– D. Sullivan. *Some remarks on quantum topology.*
– A. Verjovsky. *Complex Kleinian groups in higher dimensions.*

Finally, two parallel evening sessions gave many active researchers an opportunity to present their recent results in the field.

**Acknowledgments.** The conference was funded by the Rosenbaum Foundation, and NSF grant DMS-9805524. The other members of the organizing committee – Étienne Ghys, Yakov Eliashberg, Tony Phillips, Dennis Sullivan, and Alberto Verjovsky– were instrumental in setting the framework of the conference, selecting speakers and obtaining funds. Nothing of value could have happened at the conference were it not for the tireless effort of the staff at the mathematics department. Gerri Sciulli and Lucille Meci kept everything together starting many months before the meeting itself. Amy Dellorusso, Grace Hunt, and Barbara Wichard helped with many details and long hours.

Mikhail Lyubich, John Milnor, Yair Minsky
Stony Brook, July 2000

# Contents

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# Geodesic laminations on surfaces

Francis Bonahon

## Contents



1991 *Mathematics Subject Classification.* 53C25, 30F40, 57N05.
*Key words and phrases.* geodesic laminations, transverse structures,.
This research was partially supported by N.S.F. grants DMS-9504282 and DMS-9803445.

Geodesic laminations on surfaces were introduced by W.P. Thurston a little over 20 years ago, and have since been a very powerful tool in hyperbolic geometry, low-dimensional topology and dynamical systems. In particular, there are several different contexts where geodesic laminations now routinely occur. Geodesic laminations can be considered as:

- topological objects, occurring as generalizations of simple closed curves on surfaces;
- geometric objects, such as bending laminations of hyperbolic convex cores, shearing loci of earthquakes on hyperbolic surfaces, stable laminations of pseudo-Anosov diffeomorphisms, maximal stretch laminations between hyperbolic surfaces;
- interesting dynamical objects, in particular because of their connection with interval exchange maps.

This versatility can be somewhat confusing for the non-expert. To the expert, it provides interesting challenges when these very different points of view need to interact with each other. The minicourse given at the Workshop was devoted to illustrations of the above three aspects of geodesic laminations and of their interactions. Each of the three lectures focused on one of these viewpoints.

This article is similarly divided into three parts.

The first part is devoted to generalities on geodesic laminations, gives some examples, and discusses dynamically interesting transverse structures for geodesic laminations. This includes the analytic notion of a transverse Hölder distribution, and the more combinatorial notion of a transverse cocycle; these two transverse structures are later shown to be equivalent.

The second part discusses topological applications of geodesic laminations. In particular, we consider the space of measured geodesic laminations, as a completion of the space of simple closed curves on the surface. We mention the piecewise linear structure $\mathcal{ML}(S)$, and indicate how the combinatorial tangent vectors of this piecewise linear manifold have a geometric interpretation as geodesic laminations with transverse Hölder distributions.

Finally, the third part is devoted to some geometric applications of geodesic laminations. We chose to focus on geodesic laminations as bending loci of boundaries of convex cores of hyperbolic 3–dimensional manifolds, and as pleating loci of pleated surfaces. In particular, we show how the bending of a pleated surface along its pleating locus can be measured by a transverse cocycle. We also connect the bending of pleated surfaces to the rotation angle of closed geodesics in the hyperbolic 3–manifold.

We made the deliberate effort of closely following the style and structure of the minicourse. Anybody who has attended a talk where the lecturer directly reads from a preprint knows that the translation between mathematical writing and mathematical lecturing often requires a serious reorganization. However, we decided to try the experiment, hoping that this informal and pictorial[1] style will be more effective at communicating the main ideas, while the reader can always turn to the published references in the bibliography for a more rigorous treatment of these topics.

We are grateful to the referee for a careful reading of the manuscript, and for the suggestion of many improvements.

[1]Many pictures are taken from the monograph [**Bo6**] in preparation.

# PART I. THE DYNAMICAL VIEWPOINT

## Definitions and first properties

One of the usual frameworks for geodesic laminations is that of a surface $S$ with no boundary, endowed with a complete metric whose curvature is constant equal to $-1$ (namely a *hyperbolic metric*) and whose area is finite. There are possible variations of this setting, where we can allow the curvature of the metric to vary between two negative constants, and/or where the surface $S$ can be required to be compact with convex boundary. However, these variations only lead to minor technical differences, and we will restrict ourselves to this first setting.

An important consequence of the existence of such a hyperbolic metric is that $S$ is a surface of finite topological type, namely is homeomorphic to the complement of finitely many points in a compact surface. In addition, the Gauss-Bonnet formula implies that the Euler characteristic $\chi(S)$ is strictly negative. Conversely, it is well known that a surface $S$ of finite topological type and negative Euler characteristic always admits a hyperbolic metric of this type.

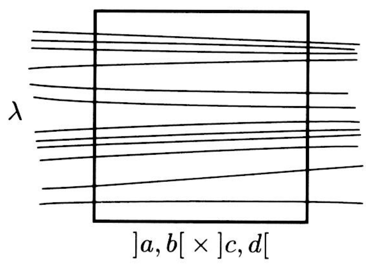


FIG. 1. The local type of a geodesic lamination

A *geodesic lamination* on the surface $S$ is a lamination $\lambda$ of $S$ whose leaves are geodesic. Recall that the fact that $\lambda$ is a lamination means that $\lambda$ is a closed subset of $S$ and is decomposed into subsets called its *leaves*, so that locally the situation is as follows: Every point of $\lambda$ has a neighborhood $U$ homeomorphic to the product $]a,b[\times]c,d[$ of two open intervals in such a way that $U\cap\lambda$ corresponds to $K\times]c,d[$ for some compact subset $K$ of $]a,b[$ and, for every leaf $g$ of $\lambda$, $U\cap g$ corresponds to $A_g\times]c,d[$ for some subset $A_g$ of $]a,b[$. Note that this local description implies that, for a geodesic lamination, the leaves are *complete* geodesics in the sense that each leaf is either closed or has infinite length in both of its ends.

In practice, checking whether a subset of $S$ is a geodesic lamination is much easier than one might think from this elaborate description of its local type. Indeed, the following property holds.

PROPOSITION 1. *If $\lambda$ is a closed subset of $S$ which is a disjoint union of simple geodesics, then $\lambda$ is a geodesic lamination.*

Here, a geodesic is *simple* if it does not cross itself. It may be closed or infinite.

The key point in the proof of Proposition 1 is the following estimate in hyperbolic geometry, whose proof can be found in [**CEG**, §**5.2.6**].

LEMMA 2. *In the hyperbolic plane $\mathbb{H}^2$, there is a constant $C$ with the following property: If the geodesic $g$ has unit tangent vector $v$ at $x$, if the geodesic $h$ has unit*

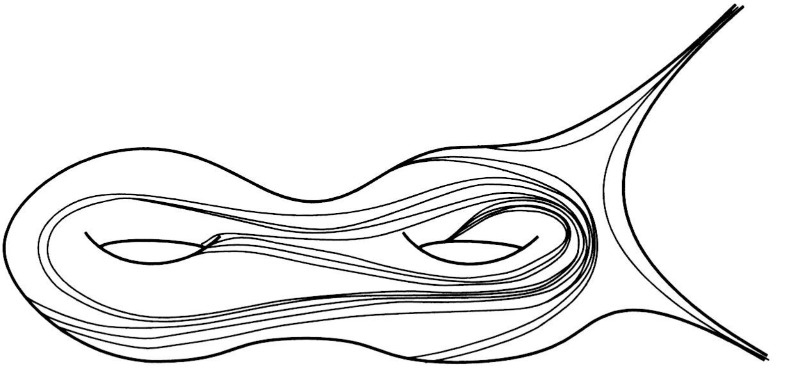

FIG. 2. A geodesic lamination

*tangent vector $w$ at $y$, and if the geodesics $g$ and $h$ are disjoint, then,*

$$d(v, \pm w) \leqslant C d(x, y).$$

In particular, if $\lambda$ is a closed subset of $S$ which is a disjoint union of simple geodesics, the tangent directions of these geodesics form a Lipschitz direction field on $\lambda$. Extending it to a Lipschitz direction field over a neighborhood of $\lambda$ and using the standard existence results for solutions to ordinary differential equations, one obtains local charts which show that $\lambda$ is a lamination, in the sense indicated above.

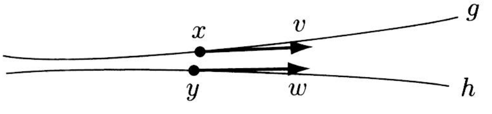


FIG. 3. Disjoint geodesics have nearby directions at nearby points

## Examples of geodesic laminations

**Geodesic laminations with finitely many leaves.** The first example is provided by a family of disjoint simple closed geodesics $\gamma_1, \gamma_2, \dots, \gamma_n$.

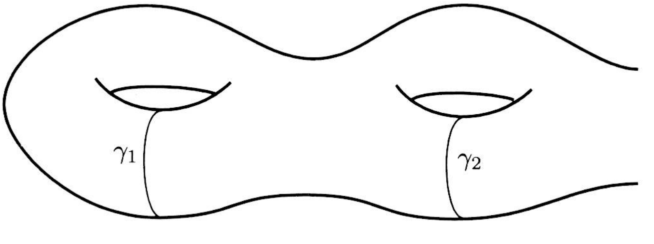


FIG. 4. A geodesic lamination with all leaves closed.

This example can be enlarged by adding a finite family of infinite geodesics which spiral along these closed geodesics, as in Figure 5.

**Interval exchange maps.** More complex examples are associated to interval exchange maps, which we now define. Consider a compact interval $I$ in $\mathbb{R}$, which we decompose as a finite union $I = I_1 \cup I_2 \cup \dots \cup I_n$ of intervals $I_i$ with disjoint

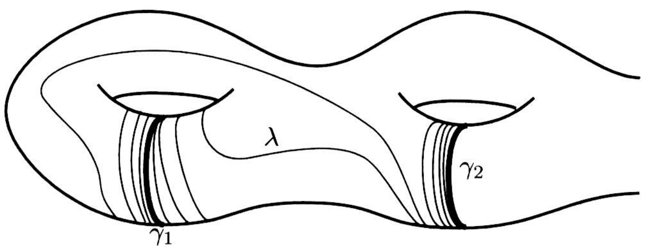


FIG. 5. A geodesic lamination with finitely many leaves.

interiors. Choose another decomposition of $I = J_1 \cup J_2 \cup \ldots \cup J_n$ into intervals $J_j$ with disjoint interiors such that, for every $i$, there is an isometry $\varphi_i : I_i \to J_i$. The collection of the $\varphi_i$ defines a 'map' $\varphi : I \to I$. This map is in general 1-to-2 at the end points of the $I_i$, but is well-defined everywhere else. Such a $\varphi$ is an *interval exchange map.*

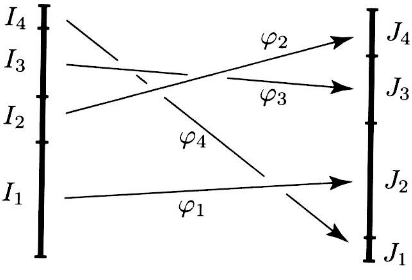


FIG. 6. An interval exchange map.

Let us restrict attention to the case where $\varphi(\partial I) \cap \partial I \neq \emptyset$. Otherwise, $\varphi$ restricts to an exchange of fewer intervals.

To the interval exchange map $\varphi : I \to I$ is associated its *suspension* $\Sigma_\varphi$, obtained by gluing $n$ rectangles $I_i \times [0,1]$ to the interval $I$ as follows: $I_i \times \{0\}$ is identified to the interval $I_i \subset I$ by the obvious map, and $I_i \times \{1\}$ to the interval $J_i \subset I$ by the (isometric) restriction $\varphi_i : I_i \to J_i$ of $\varphi$. In particular, $\Sigma_\varphi$ looks like a freeway interchange without entry or exit ramps.

Foliate each rectangle $I_i \times [0,1]$ by the arcs $\{*\} \times [0,1]$. This defines a foliation of the suspension $\Sigma_\varphi$ by lines and simple closed curves. The leaves of this foliation are in one-to-one correspondence with the 2–sided orbits of $\varphi$, namely with the bi-infinite sequences $\{\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots\}$ where $x_{n+1} = \varphi(x_n)$ for every $n \in \mathbb{Z}$ and where the indexing is only defined modulo translation in $\mathbb{Z}$. Let a leaf be *regular* if it does not pass through any of the end points of the $I_i$.

Note that the boundary of the suspension $\Sigma_\varphi$ consists of a certain number of cycles. Gluing a semi-open annulus along each of these cycles, we obtain a surface $S$ of finite type, whose Euler characteristic $\chi(S) = 1 - n$ is negative. Endow $S$ with a finite area hyperbolic metric.

Pulling tight each leaf $l$ of $\Sigma_\varphi$ defines a geodesic $g_l$ of $S$. The key technical property here is that each leaf of $\Sigma_\varphi$ is quasi-geodesic in $\Sigma_\varphi$ and (by a small argument using the fact that $\varphi(\partial I) \cap \partial I \neq \emptyset$) in $S$, so that there is a unique geodesic $g_l$

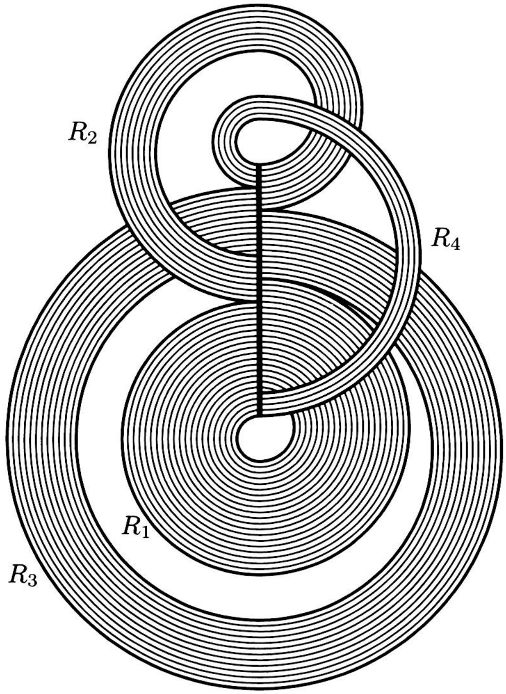


FIG. 7. The suspension of an interval exchange map.

of $S$ for which there is a homotopy from $l$ to $g_l$ which moves points by a uniformly bounded distance.

Consider the family of those geodesics $g_l$ which are associated in this way to the regular leaves of $\Sigma_\varphi$. Since distinct regular leaves $l$ and $l'$ are disjoint, the boundedness of the homotopies from $l$ to $g_l$ and from $l'$ to $g_{l'}$ shows that the associated geodesics cannot cross each other, and therefore are either disjoint or equal. Similarly, since a regular leaf does not cross itself, the geodesic $g_l$ must be simple (closed or infinite). Let $\lambda'$ be the union of the geodesics $g_l$ associated to the regular leaves $l$ of $\Sigma_\varphi$, and let $\lambda'$ be the closure of $\lambda'$. An application of Lemma 2 and Proposition 1 shows that $\lambda$ is a geodesic lamination.

If the lengths of all the intervals $I_i$ are rational, then every regular leaf of $\Sigma_\varphi$ is closed. Consequently, $\lambda$ consists of finitely many simple closed geodesics, and we are back to our first example.

The situation is much more interesting in the other extreme case where the lengths of the intervals $I_i$ are linearly independent over $\mathbb{Q}$. Then, assuming in addition that $\varphi$ does not restrict to an exchange of fewer intervals, all the regular leaves of $\Sigma_\varphi$ are infinite. In addition, two distinct regular leaves $l$, $l'$ must diverge at some point and follow different routes on the 'freeway' $\Sigma_\varphi$; it easily follows that the associated geodesics $g_l$ and $g_{l'}$ must be distinct. As a consequence, the geodesic lamination $\lambda$ so constructed now has uncountably many leaves, all infinite.

## The topology of geodesic laminations

Consider the complement $S - \lambda$ of a geodesic lamination $\lambda$ in the hyperbolic surface $S$. On the open subset $S - \lambda$, we can consider the path metric induced by the metric of $S$, for which the distance between $x$ and $y \in S-\lambda$ is the infimum of the lengths of all paths going from $x$ to $y$ in $S-\lambda$ (and is infinite if $x$ and $y$ are in different components of $S-\lambda$).

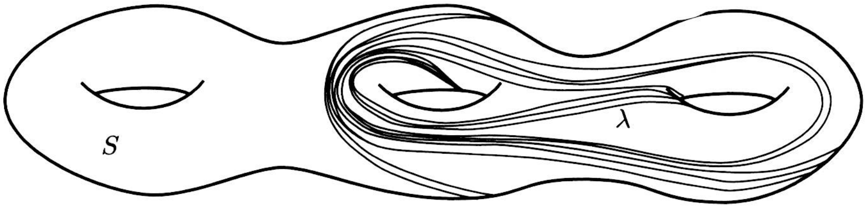


FIG. 8. A geodesic lamination $\lambda$.

Let $\widehat{S-\lambda}$ be the completion of $S-\lambda$ for this metric. From the local picture of $\lambda$, we see that $\widehat{S-\lambda}$ locally is a hyperbolic surface with totally geodesic boundary. More precisely, $\widehat{S-\lambda}$ is obtained by abstractly adding to $S-\lambda$ a boundary made up of leaves of $\lambda$.


FIG. 9. The completion $\widehat{S-\lambda}$ of the complement $S-\lambda$.

Since $S$ has finite area, it follows from the above description that $\widehat{S-\lambda}$ also has finite area. From this, we conclude that $\widehat{S-\lambda}$ is the union of a compact part, of a finite number of cusps corresponding to cusps of $S$, and of finitely many spikes, each delimited by two (possibly equal) boundary components of $\widehat{S-\lambda}$. In particular, the topological type of $\widehat{S-\lambda}$ is finite and its boundary consists of finitely many geodesics.

We can be a little more precise. Using Lemma 2, we can construct a direction field which is non-singular and is transverse to the leaves on $\lambda$, and has finitely many isolated singularities on $S-\lambda$. Using the Poincaré-Hopf formula to express the Euler characteristic $\chi(\ )$ in terms of the indices of these singularities, we conclude that

$$\chi\left(\widehat{S-\lambda}\right) = \chi(S) + \tfrac{1}{2}s$$

where $s$ is the number of spikes of $\widehat{S-\lambda}$.

A corollary of this formula is that the number of components of $\lambda$ is uniformly bounded. In particular, $\lambda$ can contain at most finitely many *sub-laminations*, namely closed subsets which are the union of leaves of $\lambda$. As a consequence, $\lambda$