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Abstract

Let S be a (discrete) semigroup, and let $\ell^1(S)$ be the Banach algebra which is the semigroup algebra of S. We shall study the structure of this Banach algebra and of its second dual.

We shall determine exactly when $\ell^1(S)$ is amenable as a Banach algebra, and shall discuss its amenability constant, showing that there are 'forbidden values' for this constant.

The second dual of $\ell^1(S)$ is the Banach algebra $M(\beta S)$ of measures on the Stone–Čech compactification βS of S, where $M(\beta S)$ and βS are taken with the first Arens product $\Box$. We shall show that S is finite whenever $M(\beta S)$ is amenable, and we shall discuss when $M(\beta S)$ is weakly amenable. We shall show that the second dual of $L^1(G)$, for G a locally compact group, is weakly amenable if and only if G is finite.

We shall also discuss left-invariant means on S as elements of the space $M(\beta S)$, and determine their supports.

We shall show that, for each weakly cancellative and nearly right cancellative semigroup S, the topological centre of $M(\beta S)$ is just $\ell^1(S)$, and so $\ell^1(S)$ is strongly Arens irregular; indeed, we shall considerably strengthen this result by showing that, for such semigroups S, there are two-element subsets of $\beta S \setminus S$ that are determining for the topological centre; for more general semigroups S, there are finite subsets of $\beta S \setminus S$ with this property.

We have partial results on the radical of the algebras $\ell^1(\beta S)$ and $M(\beta S)$.

We shall also discuss analogous results for related spaces such as $WAP(S)$ and $LUC(G)$.

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Contents

CHAPTER 1

Introduction

Our aim in this memoir is to study the algebraic structure of some Banach algebras which are defined on semigroups and on their compactifications. In particular we shall study the semigroup algebra $\ell^1(S)$ of a semigroup S and its second dual algebra; this includes the important special case in which S is a group. Here $\ell^1(S)$ is taken with the convolution product $\star$ and the second dual $\ell^1(S)''$ is taken with respect to the first and second Arens products, $\Box$ and $\Diamond$; these second dual algebras are identified with Banach algebras $(M(\beta S), \Box)$ and $(M(\beta S), \Diamond)$, which are, respectively, the right and left topological semigroups of measures defined on βS, the Stone–Čech compactification of S. We shall also study the closed subalgebras $\ell^1(\beta S)$ of $M(\beta S)$ and some related Banach algebras.

Much of our work depends on knowledge of the properties of the semigroup $(\beta S, \Box)$, and, in particular, of $(\beta \mathbb{N}, \Box)$. We wish to stress that $(\beta \mathbb{N}, \Box)$ is a deep, subtle, and significant mathematical object, with a distinguished history and about which there are challenging open questions; we hope to introduce the power of this semigroup to those primarily interested in Banach algebras. Indeed the questions that we ask about Banach algebras are often resolved by inspecting the properties of this semigroup, and sometimes require new results about it. So we also hope that those primarily interested in topological semigroups will be stimulated by the somewhat broader questions, arising from Banach algebra theory, that we raise about $(\beta S, \Box)$. In brief, we aspire to interest specialists in both Banach algebra theory and in topological semigroups in our work. For this reason we have tried to incorporate general background from each of these theories in our exposition in an attempt to make the work accessible to both communities.

This paper is partially a sequel to the earlier memoir [**21**]. (For a correction to [**21**], see p. 147 of the present work.)

Notation. We recall some notation that will be used throughout; for further details of all terms used, see [**19**] and [**21**].

We shall use elementary properties of ordinal and cardinal numbers as given in [**19**, Chapter 1.1], for example. The minimum infinite ordinal is ω and the minimum uncountable ordinal is ω_1; these ordinals are also cardinals, and are denoted by $\aleph_0$ and $\aleph_1$, respectively, in this case. The cofinality of an ordinal α is cof α; a cardinal α is *regular* if cof $\alpha = \alpha$.

The *Continuum Hypothesis* (CH) is the assertion that the continuum $\mathfrak{c} = 2^{\aleph_0}$ is equal to $\aleph_1$; this hypothesis is independent of the usual axioms ZFC of set theory. Results that are claimed only in the theory ZFC + CH are denoted by '(CH)'.

Let S be a set. The cardinality S is denoted by $|S|$, the family of all subsets of S is $\mathcal{P}(S)$, and the family of all finite subsets of S is $\mathcal{P}_f(S)$. Let κ be a cardinal.

Then

$$[S]^\kappa = \{T \subset S : |T| = \kappa\} \quad \text{and} \quad [S]^{<\kappa} = \{T \subset S : |T| < \kappa\}\,.$$

The characteristic function of a subset T of S is denoted by χ_T; we set $\delta_s = \chi_{\{s\}}$ for $s \in S$.

We set $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. The sets $\{1, \dots, n\}$ and $\{0, 1, \dots, n\}$ are denoted by $\mathbb{N}_n$ and $\mathbb{Z}_n^+$, respectively. The set of rational numbers is $\mathbb{Q}$, $\mathbb{I} = [0, 1]$, and the unit circle and open unit disc in the complex plane $\mathbb{C}$ are denoted by $\mathbb{T}$ and $\mathbb{D}$, respectively. The complex conjugate of $z \in \mathbb{C}$ is denoted by $\overline{z}$.

Algebras. Let A be an algebra (always over the complex field, $\mathbb{C}$). The product map is

$$m_A : (a, b) \mapsto ab\,, \quad A \times A \to A\,.$$

The *opposite algebra* to A is denoted by $A^{\rm op}$; this algebra has the product given by $(a, b) \mapsto ba$, $A \times A \to A$. In the case where A does not have an identity, the algebra formed by adjoining an identity to A is $A^\#$ (and $A^\# = A$ if A has an identity); the identity of A or $A^\#$ is often denoted by e_A.

The *centre* of A is

$$\mathfrak{Z}(A) = \{a \in A : ab = ba \ \ (b \in A)\}\,.$$

An *idempotent* in A is an element p such that $p^2 = p$; the family of idempotents in A is denoted by $\mathfrak{I}(A)$. For $p, q \in \mathfrak{I}(A)$, set $p \le q$ if $pq = qp = p$, so that $(\mathfrak{I}(A), \le)$ is a partially ordered set; a *minimal idempotent* in A is a minimal element of the set $(\mathfrak{I}(A) \setminus \{0\}, \le)$.

Let I be an ideal in an algebra A, and let B be a subalgebra of A such that $A = B \oplus I$ as a linear space. Then A is the *semi-direct product* of B and I, written $A = B \ltimes I$.

Let I be an ideal in an algebra A, and suppose that I has an identity e_I. Then we remark that $e_I \in \mathfrak{Z}(A)$. Indeed, for each $a \in A$ we have $ae_I, e_Ia \in I$, and so $e_I(ae_I) = ae_I$ and $(e_Ia)e_I = e_Ia$. Thus $e_I \in \mathfrak{Z}(A)$.

Let A be an algebra. We denote by R_A, N_A, and Q_A the (Jacobson) *radical* of A, the set of *nilpotent* elements of A, and the set of *quasi-nilpotent* elements of A, respectively; the sets R_A, N_A, and Q_A are defined in [**19**], but R_A is denoted by $\operatorname{rad} A$ in [**19**]. We recall that R_A is defined to be the intersection of the maximal modular left ideals of A, that R_A is an ideal in A, and that A is defined to be *semisimple* if $R_A = \{0\}$. We always have the trivial inclusions:

$$R_A \subset Q_A, \quad N_A \subset Q_A\,.$$

In general, we have $R_A \not\subset N_A$ and $N_A \not\subset R_A$; further, neither Q_A nor N_A is necessarily closed under either addition or multiplication in A. For an ideal I in A, we have $R_I = I \cap R_A$; in the case where A/I is semisimple, we have $R_I = R_A$.

A nilpotent element $a \in A$ has *index* n if $n = \min\{k \in \mathbb{N} : a^k = 0\}$; a subset S of A is *nil* if each element of S is nilpotent; the radical R_A contains each left or right ideal which is nil.

Let A be an algebra. For subsets S and T of A, we set

$$S \cdot T = \{st : s \in S,\, t \in T\}\,,$$

and $ST = \text{lin}\, S \cdot T$; we write $S^{[2]}$ for $S \cdot S$; we define S^n inductively by setting $S^{n+1} = SS^n$ $(n \in \mathbb{N})$. The set S is *nilpotent* if $S^n = \{0\}$ for some $n \in \mathbb{N}$. The algebra A *factors* if $A = A^{[2]}$.

Let A be an algebra, and let $a \in A$. Then a is *quasi-invertible* if there exists $b \in A$ such that

$$a + b - ab = a + b - ba = 0\,;$$

the set of quasi-invertible elements of A is denoted by $q\text{-Inv}A$. A subalgebra B of A is *full* if $B \cap q\text{-Inv}A = q\text{-Inv}B$. In the case where A has an identity, the set of *invertible* elements of A is denoted by $\text{Inv}A$, and the *spectrum* of an element $a \in A$ is denoted by $\sigma_A(a)$ or $\sigma(a)$, so that

$$\sigma_A(a) = \{z \in \mathbb{C} : ze_A - a \notin \text{Inv}\, A\}\,;$$

if B is a full subalgebra of A and $a \in B$, then $\sigma_B(a) = \sigma_A(a)$.

A *character* on an algebra A is a non-zero homomorphism from A onto $\mathbb{C}$; the collection of characters on A is the *character space* of A and is denoted by Φ_A. Suppose that $\Phi_A \neq \emptyset$, and take $a \in A$. Then the *Gel'fand transform* of a is defined to be $\widehat{a} \in \mathbb{C}^{\Phi_A}$, where $\widehat{a}(\varphi) = \varphi(a)$ $(\varphi \in \Phi_A)$; we define the *Gel'fand transform* of A as

$$\mathcal{G} : a \mapsto \widehat{a}, \quad A \to \mathbb{C}^{\Phi_A}\,,$$

so that $\mathcal{G}$ is a homomorphism.

Let $m, n \in \mathbb{N}$, and let S be a set. The collection of $m \times n$ matrices with entries from S is denoted by $\mathbb{M}_{m,n}(S)$, with $\mathbb{M}_n(S)$ for $\mathbb{M}_{n,n}(S)$ and $\mathbb{M}_{m,n}$ for $\mathbb{M}_{m,n}(\mathbb{C})$. In particular, $\mathbb{M}_n$ is a unital algebra; the *matrix units* in $\mathbb{M}_n$ are denoted by E_{ij}, so that

$$E_{ij}E_{k\ell} = \delta_{j,k}E_{i\ell} \quad (i, j, k, \ell \in \mathbb{N}_n)\,,$$

and the identity matrix in $\mathbb{M}_n$ is $I_n = (\delta_{i,j})$. Let A be an algebra. Then $\mathbb{M}_n(A)$ is also an algebra in the obvious way; the matrix (a_{ij}) is identified with the tensor product $\sum\{E_{ij} \otimes a_{ij} : i, j \in \mathbb{N}_n\}$, so that $\mathbb{M}_n(A)$ is isomorphic to $\mathbb{M}_n \otimes A$. If A is unital, we regard $\mathbb{M}_n$ as a subset of $\mathbb{M}_n(A)$ by identifying E_{ij} with $E_{ij} \otimes e_A$. In the case where A is commutative, the determinant, $\det a$, of an element $a \in \mathbb{M}_n(A)$ is defined in the usual way; the element a is invertible in $\mathbb{M}_n(A)$ if and only if $\det a$ is invertible in A.

Let E be a linear space. The linear span of a subset S of E is denoted by $\text{lin}\, S$. Let S and T be subsets of E. Then

$$S + T = \{s + t : s \in S,\, t \in T\}\,.$$

The linear space of all linear operators from E to a linear space F is denoted by $\mathcal{L}(E, F)$, and we write $\mathcal{L}(E)$ for the space $\mathcal{L}(E, E)$. In fact, $\mathcal{L}(E)$ is an algebra with respect to composition of operators, with identity I_E, the identity operator on E.

Let A be an algebra, and let E be an A-bimodule with respect to operations $(a, x) \mapsto a \cdot x$ and $(a, x) \mapsto x \cdot a$ from $A \times E$ to E. For subsets S of A and T of E, set

$$S \cdot T = \{a \cdot x : a \in S,\, x \in T\}\,,$$

and $ST = \text{lin}\, S \cdot T$. A map $D \in \mathcal{L}(A, E)$ is a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A)\,.$$

For $x \in E$, set

$$ad_x : a \mapsto a \cdot x - x \cdot a\,, \quad A \to E\,.$$

Then ad_x is a derivation; these are the *inner* derivations from A to E. For each $\varphi \in \Phi_A \cup \{0\}$, a *point derivation at* φ is a linear functional d on A such that

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

Let A be an algebra. Then the space $A \otimes A$ is an A-bimodule for maps that satisfy the conditions that $a \cdot (b \otimes c) = ab \otimes c$ and $(b \otimes c) \cdot a = b \otimes ca$ for $a, b, c \in A$. There is a linear map $\pi_A : A \otimes A \to A$ such that $\pi_A(a \otimes b) = ab \; (a, b \in A)$. Suppose that A is unital. Then a *diagonal* for A is an element $u \in A \otimes A$ such that $a \cdot u = u \cdot a \; (a \in A)$ and $\pi_A(u) = e_A$. For example, let $n \in \mathbb{N}$. Then

$$\frac{1}{n} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}$$

is a diagonal for $A = \mathbb{M}_n$.

Banach spaces. Let E be a Banach space. Then the closed unit ball of radius $r > 0$ in E is denoted by $E_{[r]}$. The *dual space* of E is denoted by E', and the second dual space is E''; we regard E as a closed subspace of E''. The weak-$*$ topology on E' is denoted by $\sigma(E', E)$, or simply by σ when the spaces are clear. The values of $\lambda \in E'$ at $x \in E$ and of $\Phi \in E''$ at $\lambda \in E'$ are denoted by $\langle x, \lambda \rangle$ and $\langle \Phi, \lambda \rangle$, respectively. The closure of a subset X of E' in the weak-$*$ topology is $\overline{X}^{\sigma}$. Let F be a linear subspace of E. Then the *annihilator* F° of F in E' is

$$F^{\circ} = \{\lambda \in E' : \langle x, \lambda \rangle = 0 \; (x \in F)\}.$$

Let E and F be Banach spaces. Then the space of all bounded linear operators from E to F is denoted by $\mathcal{B}(E, F)$; $\mathcal{B}(E, F)$ is a Banach space with respect to the operator norm. We write $\mathcal{B}(E)$ for $\mathcal{B}(E, E)$; the *dual* of $T \in \mathcal{B}(E, F)$ is denoted by $T' \in \mathcal{B}(F', E')$.

We use the notations c_0, ℓ^{∞}, ℓ^p for standard Banach spaces of sequences on $\mathbb{N}$ (where $p \geq 1$), and, for example, we write ℓ^p_n for the space $\mathbb{C}^n$ with the ℓ^p-norm.

The Banach space $\ell^1(S)$. Let S be a non-empty set, and consider the Banach space $\ell^1(S)$. A generic element of $\ell^1(S)$ has the form

$$f = \sum \{\alpha_s \delta_s : s \in S\},$$

where

$$\|f\|_1 = \sum \{|\alpha_s| : s \in S\} < \infty.$$

The dual space of $E := \ell^1(S)$ is $E' = \ell^{\infty}(S)$, with the duality

$$\langle f, \lambda \rangle = \sum \{f(s)\lambda(s) : s \in S\} \quad (f \in E, \lambda \in E').$$

We shall later identify $\ell^{\infty}(S)$ with $C(\beta S)$ (see Chapter 5 for more details). The Banach space $\ell^1(S)$ is identified with the dual space of $c_0(S)$, with the above duality.

Let S and T be non-empty sets. The projective tensor product $\ell^1(S) \widehat{\otimes} \ell^1(T)$ is identified with $\ell^1(S \times T)$ by setting

$$(f \otimes g)(s, t) = f(s)g(t) \quad (s \in S, \, t \in T)$$

for $f \in \ell^1(S)$ and $g \in \ell^1(T)$. Note that, for an element $F = \sum \alpha_{ij} \delta_{(s_i, t_j)}$ in $\ell^1(S) \widehat{\otimes} \ell^1(T)$, where $\{(s_i, t_j) : i, j \in \mathbb{N}\}$ is a set of distinct points in $S \times T$, we have

$$\|F\|_{\pi} = \|F\|_1 = \sum_{i,j=1}^{\infty} |\alpha_{ij}|. \tag{1.1}$$

Continuous functions and measures. Throughout, a locally compact space is assumed to be Hausdorff, unless we say otherwise.

Let Ω be a non-empty, locally compact space. Then we denote by $C(\Omega)$ the algebra (for the pointwise product) of all continuous functions on Ω. The *support* of $f \in C(\Omega)$ is the set

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}\,.$$

We denote by $CB(\Omega)$ the algebra of bounded, continuous functions on Ω, by $C_{00}(\Omega)$ the subalgebra of $CB(\Omega)$ consisting of functions of compact support, and by $C_0(\Omega)$ the subalgebra of $CB(\Omega)$ of functions that vanish at infinity. The *uniform norm* on Ω is denoted by $|\cdot|_\Omega$, so that $(CB(\Omega), |\cdot|_\Omega)$ and $(C_0(\Omega), |\cdot|_\Omega)$ are uniform algebras, as in [**19**]. The space of real-valued functions in $C(\Omega)$ is denoted by $C_{\mathbb{R}}(\Omega)$, etc.

The space consisting of all complex-valued, regular Borel measures on Ω is denoted by $M(\Omega)$; the space of real-valued measures in $M(\Omega)$ is $M_{\mathbb{R}}(\Omega)$, and the cone of positive measures is $M(\Omega)^+$. For a Borel subset B of Ω and $\mu \in M(\Omega)$, we denote the restriction measure by $\mu \mid B$. Let $\mu \in M(\Omega)$. Then $|\mu|$ is the total variation measure corresponding to μ, and so $|\mu| \in M(\Omega)^+$. By the Jordan decomposition theorem, each $\mu \in M(\Omega)$ is a linear combination of four measures in $M(\Omega)^+$. The *support* of μ, denoted by $\operatorname{supp} \mu$, is the complement of the maximal open subset U of Ω such that $|\mu|(U) = 0$.

The space $M(\Omega)$ is a Banach space with respect to the total variation norm $\|\cdot\|$, so that

$$\|\mu\| = |\mu|(\Omega) \quad (\mu \in M(\Omega)\,.$$

The Banach space $(M(\Omega), \|\cdot\|)$ is identified with the dual space of $C_0(\Omega)$ via the duality

$$\langle \mu,\, \lambda\rangle = \int_\Omega \lambda(s)\,\mathrm{d}\mu(s) \quad (\mu \in M(\Omega),\, \lambda \in C_0(\Omega))$$

(see [**19**]).

Let $\mu \in M(\Omega)^+$, and let K be a compact subspace of Ω. Recall that

$$\mu(K) = \inf\{\langle \mu,\, \lambda\rangle : \lambda \in C_{00}(\Omega),\, \lambda \geq 0,\, \lambda \mid K = 1\}\,. \tag{1.2}$$

In the case where Ω is a compact space, $\|\mu\| = \mu(\Omega) = \langle \mu,\, 1\rangle$, where 1 denotes the function constantly equal to 1 on Ω.

Let $x \in \Omega$. Then we identify x with $\delta_x \in M(\Omega)^+$, where δ_x is the point mass at x. It is standard that $\operatorname{lin}\{x : x \in \Omega\}$ is weak-$*$ dense in $M(\Omega)$. A measure $\mu \in M(\Omega)$ is *discrete* if there is a countable set E such that $|\mu|(\Omega \setminus E) = 0$; we identify the closed subspace of $M(\Omega)$ consisting of the discrete measures with $\ell^1(\Omega)$. A measure $\mu \in M(\Omega)$ is *continuous* if $\mu(\{x\}) = 0 \ (x \in \Omega)$; the closed subspace of $M(\Omega)$ consisting of the continuous measures is denoted by $M_c(\Omega)$. The discrete and continuous components of $\mu \in M(\Omega)$ are denoted by μ_d and μ_c, respectively; we have $\|\mu\| = \|\mu_d\| + \|\mu_c\|$ for each $\mu \in M(\Omega)$, and so $M(\Omega) = \ell^1(\Omega) \oplus M_c(\Omega)$ as an ℓ^1-direct sum of Banach spaces.

Let K be a compact subspace of Ω. Then

$$I(K) = \{\lambda \in C_0(\Omega) : \lambda \mid K = 0\}\,,$$

so that $I(K)$ is a closed ideal in $C_0(\Omega)$. We shall identify $M(K)$ with $I(K)^\circ$, so that

$$\begin{aligned} M(K) &= \{\mu \in M(\Omega) : \langle \mu,\, \lambda\rangle = 0 \quad (\lambda \in I(K))\} \\ &= \{\mu \in M(\Omega) : |\mu|(\Omega \setminus K) = 0\}\,. \end{aligned}$$

The subspace $M(K)$ is a weak-$*$ closed subspace of $M(\Omega)$. For each $\mu \in M(\Omega)$, we have $\mu \mid K \in M(K)$ and $\mu \mid (\Omega \backslash K) \in M(\Omega \backslash K)$, and $\|\mu\| = \|\mu \mid K\| + \|\mu \mid (\Omega \setminus K)\|$. Thus

$$M(\Omega) = M(K) \oplus M(\Omega \setminus K)$$

as an ℓ^1-direct sum of Banach spaces.

Let S be a non-empty set, and let $E = \ell^1(S)$, so that the dual of $\ell^\infty(S)$ is $E'' = M(\beta S)$. Set $S^* = \beta S \setminus S$, a closed subspace of βS, called the *growth* of S. We note that the relative weak-$*$ topology on $M(S^*)$ from σ is the same as the topology $\sigma(M(S^*), C(S^*))$, and that $\overline{\ell^1(S^*)}^{\sigma} = M(S^*)$.

Banach algebras. For the theory of Banach algebras, see [**19**]. For example, $\mathcal{B}(E)$ is a Banach algebra for each Banach space E, and $(C(\Omega), |\cdot|_\Omega)$ is a Banach algebra for each compact space Ω.

We wish to note a specific convention of the present memoir; it is different from that in [**19**]. A Banach algebra is an algebra A which is also a Banach space for a norm $\|\cdot\|$ and is such that

$$\|ab\| \le \|a\|\,\|b\| \quad (a, b \in A)\,.$$

Now suppose that A has an identity e_A; in distinction from [**19**] and some other sources, we do **not** require that $\|e_A\| = 1$.

We make the following presumably well-known remark about the norms of identities in Banach algebras.

PROPOSITION 1.1. *Let I be a closed ideal in a Banach algebra A. Suppose that A/I and I have identities $e_{A/I}$ and e_I, respectively, with $\|e_{A/I}\| = \alpha$ and $\|e_I\| = \beta$. Then A has an identity e_A, and*

$$\|e_A\| \le \alpha + \beta + \alpha\beta\,.$$

PROOF. Write $q : A \to A/I$ for the quotient map. We recall that $e_I \in \mathfrak{Z}(A)$. Take $\varepsilon > 0$. Then there exists $a_0 \in A$ with $q(a_0) = e_{A/I}$ and $\|a_0\| \le \alpha + \varepsilon$.

Consider the element $e = e_I + a_0 - a_0 e_I \in A$. Then $q(e) = e_{A/I}$ because $q(e_I) = 0$. For each $a \in A$, we have $ae - a = ea - a \in I$, and so

$$ae - a = e_I(ae - a) = e_I a e_I + e_I a a_0 - a a_0 e_I - e_I a = 0\,.$$

Thus e is the unique identity of A. We have $\|e\| \le \alpha + \beta + (\alpha + \varepsilon)\beta + \varepsilon$. This is true for each $\varepsilon > 0$, and so $\|e_A\| \le \alpha + \beta + \alpha\beta$. □

Let $(A, \|\cdot\|)$ be a Banach algebra. Then each maximal modular left ideal of A is closed, the radical R_A is a closed ideal in A, and A/R_A is a semisimple Banach algebra, but in general neither Q_A nor N_A is $\|\cdot\|$-closed in A. Again neither Q_A nor N_A is necessarily closed under either addition or multiplication in A. In the case where A is commutative, we have

$$N_A \subset Q_A = R_A = \bigcap\{\ker \varphi : \varphi \in \Phi_A\}\,.$$

However we may have the equality $R_A = Q_A$ even when A is not commutative. For example, let $\mathbb{S}_2$ denote the free semigroup on 2 generators, as in Example 3.42, below, and set $A = \ell^1(\mathbb{S}_2)$ with convolution product, a non-commutative Banach algebra. Then $R_A = Q_A = \{0\}$ [**19**, Theorem 2.3.14].

Let A be a Banach algebra. A *left approximate identity* in A is a net (e_α) in A such that $\lim_\alpha e_\alpha a = a \ (a \in A)$; a *bounded left approximate identity* in A is a left approximate identity (e_α) such that $\sup_\alpha \|e_\alpha\| < \infty$. Similarly, we define a [*bounded*] *right approximate identity* in A. A [*bounded*] *approximate identity* in A is a net which is both a [bounded] left approximate identity and a [bounded] right approximate identity. A Banach algebra which has a bounded left approximate identity and a bounded right approximate identity has a bounded approximate identity. By Cohen's factorization theorem [**19**, Corollary 2.9.30(i)], a Banach algebra with a bounded left approximate identity or a bounded right approximate identity factors.

The Banach algebra A is *essential* if $\overline{A^2} = A$.

Let $(A, \|\cdot\|)$ be a Banach algebra. For each $a \in A$, the spectrum $\sigma_A(a)$ of a is non-empty and compact. The *spectral radius* of a is denoted by $\nu_A(a)$ or $\nu(a)$; we have

$$\nu_A(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \sup\{|z| : z \in \sigma(a)\}.$$

Suppose that A is unital, that B is a unital closed subalgebra of A, and that $a \in B$. Then

$$\partial\sigma_B(a) \subset \sigma_A(a) \subset \sigma_B(a) \subset \widehat{\sigma_A(a)}, \tag{1.3}$$

where $\widehat{X}$ denotes the union of X and the bounded components of $\mathbb{C} \setminus X$ for a compact plane set X, and ∂X denotes the topological frontier of X with respect to $\mathbb{C}$.

Each character on a Banach algebra is continuous, and the character space Φ_A of a Banach algebra A is a locally compact space for the relative topology $\sigma(A', A)$; the Gel'fand transform

$$\mathcal{G} : A \to C_0(\Phi_A)$$

is a continuous homomorphism whenever $\Phi_A \neq \emptyset$. We always have the inclusion $R_A \subset \ker \mathcal{G}$; if, further, A is commutative, then $R_A = \ker \mathcal{G}$. When A is a commutative C^*-algebra, the Gel'fand transform is a surjection, and so $A \cong C_0(\Phi_A)$.

Let I be a closed ideal in a Banach algebra A, and let $q : A \to A/I$ be the quotient map. Then $q' \mid \Phi_{A/I} : \Phi_{A/I} \to \Phi_A$ is a continuous embedding.

Let A and B be Banach algebras. The *projective tensor product* $(A\widehat{\otimes}B, \|\cdot\|_\pi)$ of A and B is defined on [**19**, p. 165]; $(A\widehat{\otimes}A, \|\cdot\|_\pi)$ is itself a Banach algebra. The map $\pi_A : A \widehat{\otimes} A \to A$ is a continuous linear map such that

$$\pi_A(a \otimes b) = ab \quad (a, b \in A);$$

π is a homomorphism of algebras when A is commutative.

Let E be a Banach space. Then we regard $\mathbb{M}_{m,n}(E)$ as a Banach space by taking the norm to be specified by

$$\|(x_{ij})\| = \sum\{\|x_{ij}\| : i \in \mathbb{N}_m,\, j \in \mathbb{N}_n\} \quad ((x_{ij}) \in \mathbb{M}_{m,n}(E)). \tag{1.4}$$

In the case where A is a Banach algebra, the algebra $\mathbb{M}_n(A)$ is a Banach algebra with respect to this norm; we have $\|I_n\| = n$. However we note that this norm is different from that given in [**19**, Example 2.1.18(ii)]; if we identify $\mathbb{M}_n$ with $\mathcal{B}(\ell_n^p)$, say, where $p \geq 1$, the norm $\|\cdot\|_p$ on $\mathbb{M}_n$ is such that $\|I_n\|_p = 1$.

The Banach algebras $L^1(G)$ and $M(G)$. Let G be a locally compact group. Then we define the group algebra $L^1(G)$ (using the left Haar measure) and the