COLLECTED WORKS

of A.M. TURING

PURE MATHEMATICS

J.L. Britton
editor

NORTH-HOLLAND

Collected Works of A.M. Turing

PURE MATHEMATICS

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PREFACE

It is not in dispute that A.M. Turing was one of the leading figures in twentieth-century science. The fact would have been known to the general public sooner but for the Official Secrets Act, which prevented discussion of his wartime work. At all events it is now widely known that he was, to the extent that any single person can claim to have been so, the inventor of the "computer". Indeed, with the aid of Andrew Hodges's excellent biography, *A.M. Turing: the Enigma*, even non-mathematicians like myself have some idea of how his idea of a "universal machine" arose – as a sort of byproduct of a paper answering Hilbert's *Entscheidungsproblem*. However, his work in pure mathematics and mathematical logic extended considerably further; and the work of his last years, on morphogenesis in plants, is, so one understands, also of the greatest originality and of permanent importance.

I was a friend of his and found him an extraordinarily attractive companion, and I was bitterly distressed, as all his friends were, by his tragic death – also angry at the judicial system which helped to lead to it. However, this is not the place for me to write about him personally.

I am, though, also his legal executor, and in fulfilment of my duty I have organised the present edition of his works, which is intended to include all his mature scientific writing, including a substantial quantity of unpublished material. The edition will comprise four volumes, i.e.: *Pure Mathematics*, edited by Professor J.L. Britton; *Mathematical Logic*, edited by Professor R.O. Gandy and Professor C.E.M. Yates; *Mechanical Intelligence*, edited by Professor D.C. Ince; and *Morphogenesis*, edited by Professor P.T. Saunders.

My warmest thanks are due to the editors of the volumes, to the modern archivist at King's College, Cambridge, to Dr. Arjen Sevenster and Mr. Jan Kastelein at Elsevier (North-Holland), and to Dr. Einar H. Fredriksson, who did a great deal to make this edition possible.

P.N. FURBANK

ALAN MATHISON TURING – CHRONOLOGY

1912 Born 23 June in London, son of Julius Mathison Turing of the Indian Civil Service and Ethel Sara née Stoney
1926 Enters Sherborne School
1931 Enters King's College, Cambridge as mathematical scholar
1934 Graduates with distinction
1935 Is elected Fellow of King's College for dissertation on the Central Limit Theorem of Probability
1936 Goes to Princeton University where he works with Alonzo Church
1937 (January) His article "On Computable Numbers, with an Application to the Entscheidungsproblem" is published in *Proceedings of the London Mathematical Society*
Wins Procter Fellowship at Princeton
1938 Back in U.K. Attends course at the Government Code and Cypher School (G.C. & C.S.)
1939 Delivers undergraduate lecture-course in Cambridge and attends Wittgenstein's class on Foundations of Mathematics
4 September reports to G.C. & C.S. at Bletchley Park, in Buckinghamshire, where he heads work on German naval "Enigma" encoding machine
1942 Moves out of naval Enigma to become chief research consultant to G.C. & C.S.
In November sails to USA to establish liaison with American codebreakers
1943 January–March at Bell Laboratories in New York, working on speech-encypherment
1944 Seconded to the Special Communications Unit at Hanslope Park in north Buckinghamshire, where he works on his own speech-encypherment project *Delilah*
1945 With end of war is determined to design a prototype "universal machine" or "computer". In June is offered post with National Physical Laboratory at Teddington and begins work on ACE computer
1947 Severs relations with ACE project and returns to Cambridge
1948 Moves to Manchester University to work on prototype computer
1950 Publishes "Computing Machinery and Intelligence" in *Mind*
1951 Is elected FRS. Has become interested in problem of morphogenesis
1952 His article "The Chemical Basis of Morphogenesis" is published in *Philosophical Transactions of the Royal Society*
1954 Dies by his own hand in Wimslow (Cheshire) (7 June)

INTRODUCTION

This is one of four volumes covering Turing's mathematical works. The other volumes are concerned with mathematical logic, computer science and mathematical biology, respectively.

This division of Turing's work into four parts is to some extent arbitrary. For example, in the present volume, the papers on matrices (1948) and on the zeta-function (1953) have strong connexions with computer science (the latter paper even including a technical description of hardware), while the papers on the word problem (1950 and II) and the popular article (1954) might have been included in the volume on mathematical logic. And of course, Turing's masterpiece, *On Computable Numbers, with an Application to the Entscheidungsproblem* (1937) could equally well be classified as mathematical logic or computer science.

The excellent biography of Turing by Andrew Hodges (HODGES 1983) may profitably be read in conjunction with the four volumes.

One could say that Turing's first published work, though with limited circulation, was his fellowship dissertation of 1935. It is interesting that this was on statistics. Later, he always looked out for any statistical aspects of the problem under consideration. Elementary statistics occurs, for instance, in the papers (1948, III and V).

Moreover, he made further contributions to statistics. These are described in this volume by Professor I.J. Good, who for a period during the war was Turing's main statistical assistant. This statistical section is divided into two parts: his paper of 1979 on Turing's statistical work and an introduction to that paper, specially written for this volume. This throws some interesting new light on the work during the war on cryptanalysis.

The paper on permutation groups (III) is also related to cryptanalysis; its terminology indicates that it was motivated by a study of the Enigma machine (cf. the biography). This is perhaps the most interesting of the unpublished papers from today's standpoint.

The originals of the unpublished papers (I)–(IV) and the dissertation are in the archives at King's College, Cambridge. I have been unable to locate the original of (V) although Note (3.40) of the biography suggests that this is also at King's College, Cambridge, possibly due to a confusion between (IV) and (V), both of which are on the distribution of prime numbers. However, I had access to a photocopy of (V); unfortunately, in some places not everything on the page has been copied.

The originals are in many places hard to decipher, except for the dissertation in which the handwriting is quite neat. I made hand copies of (I)–(IV) and then a typed version of these for the printers.

We now briefly review the papers by Turing included in this volume and the two related papers by BOONE (1958) and by COHEN and MAYHEW (1968). Technical summaries of Turing's longer papers appear later in the volume.

Equivalence of Left and Right Almost Periodicity (1935)

Turing called the result of this paper a "small-scale discovery" but it was surely a promising beginning to have noticed something that Von Neumann, already enormously successful, had missed.

The topic of almost periodicity has various aspects but the one being considered here is concerned with complex-valued functions on an abstract group.

This is a short and easily readable paper.

Finite Approximations to Lie Groups (1938 A)

Let G be an abstract group and a metric space such that the group operations (product and inverse) are continuous. It is known that without loss of generality the metric can be chosen such that

$$D(ax, ay) = D(x, y),$$

for all x, y, a in G. Call G *approximable* if to each $\varepsilon > 0$ we have a finite group H_ε which is a subset, not in general a subgroup, of G and (i) each x in G is within distance ε of some element of H_ε; (ii) if a, $b \in H_\varepsilon$, then $D(a \circ b, ab) < \varepsilon$, where $a \circ b$ is the product in H_ε and ab is the product in G.

S. Ulam had proposed the problem: Which groups are approximable?

In this paper Turing shows that if a connected Lie group is approximable, then it is compact and Abelian.

The proof is interesting; it involves an interplay between representations of finite groups and representations of compact groups.

The Extensions of a Group (1938 B)

Recall that a group G is an *extension* of a group N by a group B if N is a normal subgroup of G and G/N is isomorphic to B. Let A be the group of automorphisms of N and I the group of inner automorphisms. Then there is a homomorphism $\theta : G/N \to A/I$ given by $gN \to \hat{g}I$ $(g \in G)$, where $\hat{g}$ is the automorphism of N defined by $\hat{g}(n) = g^{-1}ng$ $(n \in N)$.

Now assume we are given a group N, a group B and a homomorphism $X: B \to A/I$. When does an extension G of N by B exist such that, if the isomorphic groups G/N and B are identified, then θ is just X? This is the problem considered in this paper.

This was an important contribution to the theory of extensions of a not necessarily Abelian group as it stood at that time.

A Method for the Calculation of the Zeta-function (1943)

Some Calculations of the Riemann Zeta-function (1953)

It is probably not widely known that Turing did some important theoretical work on the Riemann zeta-function besides his computational work on this function.

Recall that the Riemann zeta-function $s \to \zeta(s)$, $s = \sigma + it$, is analytic in $\mathbb{C}$ except for a pole at $s = 1$. When $\sigma > 1$ we have

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad \zeta(s) = \prod (1 - p^{-s})^{-1},$$

where the product is over all prime numbers p. The function has infinitely many zeros in the strip $0 < \sigma < 1$ while outside the strip its only zeros are $s = -2, -4, -6, \ldots$. The Riemann hypothesis (conjecture) is that all the zeros in the strip lie on the line $\sigma = \frac{1}{2}$.

The objective of the first paper (written in 1939 but not published until 1943) is to obtain a practical method for calculating $\zeta(\frac{1}{2} + it)$. (There is an elaborate piece of contour integration, done with bravura.)

The other paper (1953) describes how in 1950 the Manchester University computer was used to investigate the zeros of the zeta-function for $2\pi(63)^2 \leqslant t \leqslant 2\pi(64)^2$ and $0 < t < 1540$ in the hope of finding zeros (in the strip) off the critical line. The first part of this paper gives the theoretical basis of the method while the second part gives a brief description of the hardware and states some of the practical computing strategy. It gives some insight into the excitement and frustrations of the early days of computing.

Thanks are due to Dr. D.R. Heath-Brown for an assessment of Turing's work on the zeta-function; this appears later, after the technical summary and notes on the 1953 paper. (See also EDWARDS 1974.)

Rounding-off Errors in Matrix Processes (1948)

This is an enjoyable paper on numerical analysis with Turing "thinking aloud'' about solving linear equations and inverting matrices, starting from scratch.

The interest is on inverting $n \times n$ matrices when n is large and whether, in carrying out any of the standard methods for inversion, errors build up exponentially with n. Consideration is given to statistical bounds for errors but the emphasis is on absolute bounds.

My colleague Frank D. Burgoyne says: ''Turing's paper was one of the earliest attempts to examine the error analysis of the various methods of solving linear equations and inverting matrices. His analysis was basically sound. The main importance of the paper was that it was published at the dawn of the modern computing era, and it gave indications of which methods were ''safe'' when solving such problems on a computer.''

The Word Problem in Semi-groups with Cancellation (1950)

A *semi-group* is a set within which is defined an associative product. A *semi-group with cancellation* (SWC) is a semi-group S such that for all a, b, c in S $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$. Let K denote ''semi-group, SWC or group''. A *K-presentation* is a pair (S, D), where S is a finite set of symbols $s_1, \ldots, s_n$ and D is a finite set of formal equations $U_i = V_i$ $(i = 1, \ldots, r)$ where U_i and V_i are words in the symbols; a *word* means a finite string of symbols in the semi-group or SWC case, but means an expression of the form $s_{i_1}^{e_1} \ldots s_{i_k}^{e_k}$ where $e_j = \pm 1$ $(j = 1, \ldots, k)$ in the group case. We say that $W_1 = W_2$, where W_1, W_2 are words, is a *relation* for (S, D) if, whenever X is a K containing elements $s_1, \ldots, s_n$ such that $U_i = V_i$ in X for all i, then $W_1 = W_2$ in X. In particular each $U_i = V_i$ is called a *defining relation* (or fundamental relation (F.R.)).

We say that the *word problem is solvable for* (S, D) if there is an algorithm which will determine of any pair of words W_1, W_2 whether or not $W_1 = W_2$ is a relation for (S, D).

For any (S, D) one can construct $[S, D]$, which is a K and which is unique up to isomorphism; it contains elements $s_1, \ldots, s_n$, each element of it is a word in $s_1, \ldots, s_n$ and $W_1 = W_2$ is a relation for (S, D) if and only if we have $W_1 = W_2$ in $[S, D]$.

In 1947, Post and Markov independently showed that there is a semi-group presentation with unsolvable word problem. The problem of extending this result to groups received a lot of attention but proved difficult. In the present paper, Turing considers the half-way house of semi-groups with cancellation; undoubtedly it influenced both Novikov who finally obtained the result for groups in 1955 and Boone, who proved the result independently at about the same time.

Solvable and Unsolvable Problems (1954)

This is an article from the Penguin *Science News* which would attract the same kind of reader as, say, the *Scientific American* but which was in paperback form and was published quarterly. The article is an enjoyable and successful piece of popular exposition.

A Note on Normal Numbers (I)

Normal numbers are defined as follows. Let α be a real number and let $t \geqslant 2$ be an integer. Consider the expansion of α in scale t and let the part after the decimal point be $.\alpha_1\alpha_2\alpha_3\dots$. Let γ be an ordered set of numbers from $\{0, 1, \dots, t-1\}$ and let $l(\gamma)$ be the number of elements in γ. Denote by $S(\alpha, t, \gamma, R)$ the number of occurrences of the block γ in $\alpha_1\alpha_2\dots\alpha_R$. Then α is *normal* if

$$R^{-1}S(\alpha, t, \gamma, R) \to t^{-r} \quad \text{as } R \to \infty,$$

for all γ, t, where $r = l(\gamma)$.

It is known that almost all numbers are normal but it is not easy to give an example of one. In this unpublished paper Turing discusses the construction of normal numbers.

The Word Problem in Compact Groups (II)

As mentioned earlier, if (S, D) is a group presentation, the corresponding group $[S, D]$ may have unsolvable word problem. If, however, we restrict attention to presentations such that $[S, D]$ is a compact group, is the word problem always solvable? Turing answers this in the affirmative by making use of an important theorem of Tarski (1948) in mathematical logic.

On Permutation Groups (III)

Although this paper was evidently motivated by Turing's study of the Enigma machine (see Hodges 1983), it is essentially an important piece of pure mathematics. Turing was led to consider what turns out to be a formidable problem on permutation groups, which is as follows.

Consider permutations of the objects $a_1, \dots, a_T$. Let R be the T-cycle $(a_1\, a_2 \dots a_T)$. For any permutation U, let $H(U)$ denote the group of all permutations of the form

$$R^{t_0}UR^{t_1}\dots UR^{t_p}, \quad \sum t_i = 0.$$

$H(U)$ is called *exceptional* if it is not the symmetric or alternating group. The problem is to find all exceptional groups or at least to find all U such that $H(U)$ is exceptional.

Besides employing his usual ingenuity, Turing has to perform some really extensive calculations in order to solve the problem for the cases $T=1,2,\ldots,8$.

Clearly this problem is a challenge to present day workers in permutation groups. Computer scientists also may find it interesting to see if they can check Turing's results and extend his calculations beyond $T=8$.

The Difference $\psi(x)-x$ (IV)

Recall that Chebyshev's ψ-function is defined by $\psi(x)=\sum \log p$, where the sum is over all prime numbers p and positive integers m such that $p^m \leqslant x$. We have $\psi(x) \sim x$ as $x \to \infty$; this is equivalent to the prime number theorem that $\pi(x) \sim x/\log x$ as $x \to \infty$, where $\pi(x)$ is the number of primes less than or equal to x. In this paper it is shown that for some x ($\exp 99 < x < \exp 10^{428}$) $(\psi(x)-x)/x^{1/2} > 1.0001$. It is known that $\psi(x)-x$ changes sign infinitely often.

A.M. Turing and S. Skewes, *On a Theorem of Littlewood* (V)

In spite of the joint authorship, it seems probable that this unpublished paper was written by Turing alone.

Let $\pi(x)$ denote as usual the number of primes $\leqslant x$ and let

$$\operatorname{li} x = \lim_{h \to 0+} \left(\int_0^{1-h} + \int_{1+h}^{x} \right) \left(\frac{1}{\log u} \right) du.$$

We have $\pi(x) \sim \operatorname{li} x$ as $x \to \infty$. The Riemann hypothesis is equivalent to $\pi(x) - \operatorname{li} x = O(x^{1/2} \log x)$ as $x \to \infty$, but here the interest is on the sign of $\pi(x) - \operatorname{li} x$. Littlewood proved that

$$\frac{\pi(x) - \operatorname{li} x}{(x^{1/2}/\log x) \log\log\log x}$$

has positive limit superior and negative limit inferior as $x \to \infty$; thus $\pi(x) - \operatorname{li} x$ changes sign infinitely often. It is negative for $2 < x < 10^7$. The main part of this paper is devoted to a proof that $\pi(x) - \operatorname{li} x$ is positive for some x ($2 < x < \exp\exp 661$).

The second half of the paper, which was undoubtedly written by Turing alone, contains some informal and speculative passages as well as a substantial theorem. Attention here centres on the consequences for the theory of $\pi(x) - \operatorname{li} x$ if an actual zero of the zeta-function off the critical line were found by means of a computer.

W.W. Boone, *An Analysis of Turing's "The Word Problem in Semi-groups with Cancellation"*, (1958)

This is a careful study of Turing's paper (1950) and the author confirms that Turing's proof is essentially correct.

A.M. Cohen and M.J.E. Mayhew, *On the Difference* $\pi(x) - \operatorname{li} x$, (1968)

This is a considerably corrected and amplified version of the first part of the TURING and SKEWES paper (V). They show that the proof in that paper is essentially correct but the conclusion is that $\pi(x) - \operatorname{li} x$ is positive for some x, $2 < x < \exp\exp 1236$ (rather than $\exp\exp 661$).

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Thanks are due first to the authors and publishers who gave their permission to reprint the papers in this volume and to King's College, Cambridge, for allowing access to the unpublished papers.

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Finally my thanks are due to Jack Good for agreeing to write a special article for this volume and, on his behalf, I thank Donald Michie and Shaun Wylie.

J.L. BRITTON
19th May, 1989

POSTSCRIPT

I was a research student at Manchester University from September 1951 to August 1953. Turing's position was Deputy Director of the Computing Machine Laboratory; thus he was not formally a member of the mathematics staff but in practice he was a star member of it. The department was a very strong one: M.H.A. Newman, M.J. Lighthill, M.S. Bartlett, Kurt Mahler, G.I. Camm, B.H. Neumann, C.R. Illingworth, F.G. Friedlander, Walter Ledermann, Graham Higman, H.G. Hopkins, G.E.H. Reuter, A.H. Stone, Eric Wild, Samuel Levine, D.S. Jones, P.J. Hilton, J.E. Moyal, F.D. Kahn, G.E. Wall, J.A. Green, M.B. Glauert, Yael N. Dowker, R.K. Livesley and A.M. Walker.

I was introduced to Turing by my supervisor, Bernhard Neumann, in 1951. I can recall little of Turing. It was only towards the end of my stay that I first became interested in his work.

Turing did not give any lecture courses. I was friendly with his research student, Ivor Jones, whose field was logic.

The only lecture by Turing that I attended was one he gave to the student mathematical society. It was entitled "On large numbers". Figuring in the lecture were the following large numbers M and N. A hypothetical bird flies, once each year, to the top of Mount Everest and removes one grain of "sand"; M is the number of seconds needed to level the mountain. The other number N was such that $1/N$ is the probability that "this piece of chalk will jump from my hand and write a line of Shakespeare on the board before falling to the ground". Turing had, of course, numerical estimates for M and N.

Bernhard Neumann told me the following story. At one time (probably 1949) he believed he had discovered a proof of the solvability of the word problem for groups; upon mentioning this to Turing, he was disconcerted to learn that Turing had just completed a proof of its unsolvability. Both of them urgently re-examined their proofs and both proofs were found to be wrong.

J.L.B.

REMARKS ON TURING'S DISSERTATION

Turing's Fellowship dissertation was circulated to the electors of fellowships at King's College, Cambridge, namely Messrs. Ingham, Keynes, Braithwaite, Matthews, McCombie, Beves, F.L. Lucas and the Provost. The date of the election was March 16th, 1935. According to the biography, Philip Hall and A.C. Pigou also acted as electors.

The dissertation was entitled "On the Gaussian error function". It contains a proof of the Central Limit Theorem, which Turing discovered without knowing that a proof already existed; it was originally proved by Lindeberg in 1922.

Although the dissertation is in immediately publishable form, it seems inappropriate to disturb its present status by reprinting it in this volume. For biographical interest, however, the preface is reprinted below.

Preface

The object of this paper is to give a rigorous demonstration of the "limit theorem of the theory of probability". I had completed the essential part of it by the end of February 1934 but when considering publishing it I was informed that an almost identical proof had been given by Lindeberg[1]. The only important differences between the two papers is that I have introduced and laid stress on a type of condition which I call quasi-necessary (§8). We have both used "distribution functions" (§2) to describe errors instead of frequency functions (Appendix B) as was usual formerly. Lindeberg also uses (D) of §12 and Theorem 6 or their equivalents.

Since reading Lindeberg's paper I have for obvious reasons made no alterations to that part of the paper which is similar to his (viz. §9 to §13), but I have added elsewhere remarks on points of interest and the appendices.

So far as I know the results of §8 have not been given before. Many proofs of the completeness of the Hermite functions are already available (footnote p.33) but I believe that that given in Appendix A is original. The remarks in Appendix B are probably not new. Appendix C is nothing more than a rigorous deduction of well-known facts. It is only given for the sake of logical completeness and it is of little consequence whether it is original or not.

[1]*Math. Z.* **15** (1922).

My paper originated as an attempt to make rigorous the "popular" proof mentioned in Appendix B. I first met this proof in a course of lectures by Prof. Eddington. Variations of it are given by Czüber, Morgan, Crofton and others. Beyond this I have not used the work of others or other sources of information in the main body of the paper, except for elementary matter forming part of one's general mathematical education, but in the appendices I may mention Liapounoff's papers which I discuss there.

I consider §9 to §13 is by far the most important part of this paper, the remainder being comment and elaboration. At a first reading therefore §8 and the appendices may be omitted.