Metric Spaces

Second Edition

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Narosa

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PREFACE TO THE SECOND EDITION

Based on suggestions received from the readers, this edition has been thoroughly revised to include:

- The concept of neighborhood in the beginning, open sets and interior points are defined via neighborhood. The proofs of the results relating to open sets and interior points are modified accordingly. Closed sets are defined as the complement of an open set and the necessary changes in the statements of the results and their proofs on closed sets are made.
- Presenting the concept of equivalent metrics in Chapter 2 so that it can be used in subsequent chapters.
- Motivation for homeomorphism.
- A number of results, problems, examples and counter examples along with examples of non-metric spaces at appropriate places. Some of the important problems are solved. Moreover, problems have been distributed throughout each chapter, section wise.
- Geometrical interpretation of open spheres in $\mathbb{R}^2$ and their figures.

The authors are grateful to the readers who made valuable suggestions for improvement of the book, especially to Dr. Sapna Jain of Miranda House who read the book thoroughly and, besides, pointing out several printing mistakes, gave various ideas & suggestions for the benefit of the readers. In future also, the authors will be happy to receive suggestions and comments for further improvement.

PAWAN K. JAIN
KHALIL AHMAD

PREFACE TO THE FIRST EDITION

The term metric is derived from metor (measure). The concept of a metric space is essentially due to a French mathematician Maurice Fréchet (1878-1973), through the definition presently in use is that given by the German Mathematician Felix Hausdorff (1868-1942) in 1914. Fréchet introduced the notion in his doctoral thesis presented to the University of Paris in 1906 and for many years pioneered the study of such spaces and their applications to other areas of mathematics. It was toward the end of the last century that mathematicians, namely, Klein, Hilbert began to appreciate the power of generalised methods such as those represented by the study of metric spaces, topological spaces, groups, rings, categories etc. which have proved central to twentieth century mathematics.

Now-a-days, the study of metric spaces is considered fascinating and highly useful because of its increasing role in mathematics and applied sciences. It has been increasingly realised that this branch of mathematics is a convenient and very powerful way of examining the behaviour of various mathematical models, and it clarifies, regresses and unifies the underlying concepts in mathematics, engineering, theoretical physics, applied mathematics, economics and other applied fields.

It is in the interest of the students of mathematics, in particular, and applied, basic and social sciences, in general, that they become acquainted with the basic ideas of metric spaces in their early studies. With this idea in mind one semester course of this aspect has been prescribed in almost all the universities in India and abroad either at the undergraduate or graduate level.

Numerous books are available on topology which contain metric spaces only as a chapter, but we felt that no book could be used systematically as a textbook for an elementary course in metric spaces, particularly when the course in metric spaces is given at the undergraduate level. Some of the existing books on metric spaces do not correspond with the level of undergraduate and first year graduate students in the developing countries and in Indian universities and while others may not have the choice of the topics we would like to see these in our universities. This book is aimed to serve as a textbook for an introductory course in metric spaces for the senior undergraduate and graduate students. It can also the read with great interest by senior students of applied mathematics, statistics, engineering, physics and theoretical scientists. However, such students may omit the proofs of some of the lengthy and difficult theorems.

The text has been arranged by sections, spread out in seven chapters and two appendices beginning with a chapter on preliminaries and providing a sequence of definitions and theorems which we hope will illustrate the main ideas required in the development of the book. The purpose of this chapter is to establish a uniform notation and cover the background material in real analysis and elementary results in set theory, mappings, relations etc. It is followed by chapters 2 to 7 which contain the main material on Metric Spaces; namely, Introductory Concepts, Completeness, Continuous Functions, Compactness, Connectedness, and Fixed Point Theorems. At the end of the text an Appendix on Cantor Set and another on Limits in Metric Spaces are given. Cantor Set has its own importance that it possesses several interesting properties which are pointed out in various chapters of the book. Since the construction of this set is not easily accessible to students at this level, it was decided to provide in detail as an appendix.

We have learnt from experience that abstract ideas must be introduced quite gradually and be based on some motivation and firm foundation from concrete examples. Thus, each chapter begins with a brief introduction which hopefully will provide inspiration and a keen desire to proceed to subsequent material. Several examples are given which will illustrate to the students the invaluable role played by metric spaces in solving concrete problems in analysis. We attempt to illustrate most of the theory by examples.

Problems, remarks and notes at places would help the students to increase their knowledge by applying previous results or by presenting new material. Some of the problems require extra efforts on the part of students. Wherever thought necessary, hints are provided for the solution. The problems graded in a proper way are presented at the end of each section.

Without claiming originality of the results we do claim simplicity and lucidity of presentation coupled with comprehensiveness of the material. The various sources that have inspired the authors are listed in the bibliography. Yet the work of Copson and Simmons have made significant contribution in making the book useful for the students. The genesis of the present text lies in the classroom notes prepared by the authors in courses of 'Metric Spaces' and 'Topology' at the University of Delhi (Delhi) and Jamia Millia Islamia (New Delhi) over a period of some fifteen years–such notes were developed, revised, rewritten and expanded more times than one can recall.

Our debt to a large number of authors who have contributed to the discipline in the past is immense. We have not tried to attribute the various theorems and proofs to their original discoverers. However, we take it as our duty to place on record our gratitude to all of them.

We are thankful to our associates in the Department of Mathematics, University of Delhi and Jamia Millia Islamia and fellow mathematicians in India. They tempered the ideas and results in the book by valuable discussions from time to time. In particular, we owe a special debt to our esteemed friend Prof. R. Vasudevan of the University of Delhi who has gone through the manuscript thoroughly and made a number of valuable suggestions. Above all, we are thankful to the generation of students who have made valuable contributions in injecting simplicity in presentation of the material so as to the intelligible to the students community in general.

To our families, we owe the moral support extended throughout the execution of this project. Finally, we would like to thank Mr. N.K. Mehra and Mr. M.S. Sejwal, Narosa Publishing House for their best cooperations in bringing out this book.

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LIST OF SYMBOLS

$\in$	belongs to
$\notin$	does not belong to
$\subset$	subset
$\subsetneq$	proper subset
$\cup$	union
$\cap$	intersection
$\forall$	for all
$\exists$	there exists
$\Rightarrow$	implies
$\Leftrightarrow$	implies and is implied by, if and only if
$\{\}$	set
ϕ	empty set
A–B	difference
A^c	complement of *A*
A^o	interior of *A*
A'	set of all limit points of *A*
$\bar{A}$	closure of *A*
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
C	Cantor set
[a, b]	closed interval
]a, b[	open interval
$f: A \rightarrow B$	function from *A* to *B*
I	identity function
$g \circ f$	composite function
$f\|E$	restriction of *f* to the set *E*
$d(x, y)$	distance between *x* and *y*
$d(x, A)$	distance of *x* from the set *A*
$d(A, B)$	distance between sets *A* and *B*

$d(A)$	diameter of A
nbd	neighbourhood
$\inf A$	infimum (or greatest lower bound) of A
$\sup A$	supremum (or least upper bound) of A
$S_r(x)$	open sphere with radius r and centre x
$S_r[x]$	closed sphere with radius r and centre x
$\mathbb{R}_u$, $\mathbb{C}_u$	usual metric spaces
$\mathbb{R}^n$	Euclidean n-space
$\mathbb{C}^n$	unitary n-space
l^p	a sequence space
l^∞	space of bounded sequences
$B[a, b]$	space of functions defined and bounded on [a, b]
$C[0, 1]$	space of continuous functions on [0, 1]
$C[a, b]$	space of continuous functions on [a, b]
$\mathscr{P}[a, b]$	set of polinomials on [a, b]
$\mathscr{R}[0, 1]$	space of absolutely integrable functions in sense of Riemann

SCHEME OF NUMBERING

* Chapters are numbered by Arabic numerals.
* Sections are numbered in two digits. The first digit indicates the chapter while the second one the section of the chapter.
* All items such as Definitions, Examples, Lemmas, Theorems, Corollaries etc., irrespective of the nature of the item, are numbered in three digits. The first two digits indicate the section while the third one, the item of the section.
* Equations or display are numbered serially, within small parenthesis, for each chapter.
* Diagrams are numbered serially for each section.
* Problem sets are given at the end of each section and are numbered serially for each chapter.

CROSS-REFERENCING

* A reference to Example 4 of Examples 2.1.2 is made as Example 4 within the Examples 2.1.2 and Example 2.1.2(4) elsewhere.
* A reference to Problem 5 of Chapter 3 is made as Problem 5 within Chapter 3 and as Problem 3.5 in other chapters.

CONTENTS

1

PRELIMINARIES

This chapter is to help the reader in reviewing the preliminaries needed subsequently in this book. It is presumed that the reader has pursued an elementary course in set theory. The approach adopted in this chapter is somewhat different from that used in other chapters. It is descriptive and the arguments given are directly towards plausibility and understanding rather than rigorous proof. The preliminaries are divided into six sections.

1.1 SETS

A *set* is a well defined collection or system of objects. Other words such as collection, class and aggregate are used synonymously for the term set. 'Well defined' means that it is possible to determine readily whether an object is a member of a given set or not. The objects that belong to a set are called its *elements* (or *points* or *members*). If A is a set, then $a \in A$ denotes that a is an element of A and the notation $a \notin A$ denotes the negation of $a \in A$. For any element a and a set A, either $a \in A$ or $a \notin A$.

Two methods used frequently to describe sets are the '*tabulation method*' and the '*defining property method*'. The first, the ta ulation method, enumerates or lists the individual elements separated by commas and enclosed in braces. By this method, the set of vowels of English alphabets is written as (a, e, i, o, u). Sets which are difficult to describe by an enumeration are described by the second method–the definining-property method. In fact, this method is often more compact and convenient. A defining property of a set is a property which is satisfied by each element of that set and by nothing else. This standard notation for a set so described is $\{x|\}$ or $\{x:\}$. Here x is a dummy symbol and the space between : and } is filled by a defining-property. The above set, by this method is described as $\{x : x$ is a vowel of English alphabets).

Given two sets A and B, if the relation $a \in A$ implies $a \in B$ for all a, we say that A is a *subset* of B (or B is a *superset* of A, or A is contained in B or B contains A). In symbol, it is written as $A \subset B$ or $B \supset A$. Two sets A and B are equal if $A \subset B$ and $B \subset$ A. Generally, a set is completely determined by its elements but there is a set which has no elements, and we call it as the *empty* (or *void* or *null*) set and denote it by ϕ (phi). If A is any set, each element of ϕ (there are none) is an element of A, and so $\phi \subset A$. Thus, the empty set is a subset of every set. Further, if A is a subset of B with $A \neq \phi$ and $A \neq B$, then A is a *proper subset* of B (or B properly contains A). In other words, a set A is a proper subset of B if and only if $a \in A$ implies $a \in B$, and there exists at least one $b \in B$ such that $b \notin A$ and $A \neq \phi$.

Let A be a set. Then, the collection of all subsets of A is called the *power set* of A and is denoted by $\mathscr{P}(A)$. For instance, if A is a set containing a, b and c as its elements, there are eight subsets of A. Hence, the power set $\mathscr{P}(A)$ would contain eight elements, each being a subset of A. It is obvious that the sets ϕ and A are always members of $\mathscr{P}(A)$. In particular, $\mathscr{P}(A)$ is always a non-empty set. If A is a finite set containing n (distinct) elements, $\mathscr{P}(A)$ has 2^n elements and this is the reason for the name 'power set.'

Let A and B be two sets. Using certain operations on A and B, we can obtain four other sets. One of these is called the *union* of the two sets: written $A \cup B$ (sometimes, called the *sum* and written as $A + B$); it consists of all elements that are in A or in B or in both (an element that is in both is counted once). The second is called the *intersection* of two sets, written $A \cap B$ (sometimes, called the *product* and written as $A \,.\, B$): it consists of all elements in A as well as in B. The third one is called the *difference* of the two sets, written $A - B$: it consists of all those elements of A which are not elements of B. The fourth one is called the *cartesian product* of the two sets; written $A \times B$, it consists of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Two sets A and B are said to be *disjoint* if $A \cap B = \phi$, otherwise A intersects B. If $B \subset A$, $A - B$ is called the *complement* of B with respect to A. In case A is taken as a universal set, $A{-}B$ is written as B^c (or $\sim B$) and simply read as complement of B. If $A = \mathbb{R}$, the set of all real numbers, then $\mathbb{Q}^c$ (complement of $\mathbb{Q}$, the set of all rational numbers) is the set of all irrational numbers.

Let A, B and C be any three non-empty sets. Then, the following laws hold good;

1. **Commutative laws**
   $A \cup B = B \cup A$ and $A \cap B = B \cap A$
2. **Associative laws**
   $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
3. **Distributive laws**
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and
   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. **De Morgan's laws**
   $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

A Set whose elements are used as names is called an *index set*. An index set may be finite or infinite. Suppose for each member α of a fixed set Λ, we have a set A_α. Then Λ is the index set and the sets A_α are called the *indexed sets*, and the subscript α of A_α i.e. each $\alpha \in \Lambda$, is called an *index*. The collection of sets A_α is called a *family of indexed sets* and is denoted by $\{A_\alpha\}_{\alpha \in \Lambda}$. We shall be using the symbol Λ for index set throughout the book. Let X be a set and Λ an index set. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be an indexed family of the subsets of X. Then, the union of all the sets A_α is the set

$$\{x \in X : x \in A_\alpha, \text{ for some } \alpha \in \Lambda\}$$

We denote it by $\bigcup_{\alpha \in \Lambda} A_\alpha$. We may define $\bigcap_{\alpha \in \Lambda} A_\alpha$ similarly. De Morgan's laws hold good in an indexed family of sets. If $\{A_\alpha\}_{\alpha \in \Lambda}$ is an indexed family of subsets of X, then

$$\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c \text{ and } \left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^c = \bigcup_{\alpha \in \Lambda} A_\alpha^c$$

1.2 FUNCTIONS

Let A and B denote arbitrary given sets. By a *function* $f : A \to B$, we mean a rule which assigns to each element a of A, a unique element b of B. If $a \in A$, the corresponding element $b \in B$ is called the *f-image* of a and is denoted by $f(a)$, i.e. $b = f(a)$. In this case, a is called the *pre-image* of b. The set A is called the *domain* of the function f, and B the *co-domain* of f. The set $B_1 \subset B$ consisting of all f-images of elements of A is called the *range* of f, denoted by $f(A)$. A function f whose co-domain is $\mathbb{R}$ is called a *real-valued function.*

If f and g are two functions defined on the same domain A and if $f(a) = g(a)$ for every $a \in A$, the functions f and g are equal and we write $f = g$. Let f be a function of A *into* B. Then $f(A) \subset B$. If $f(A) = B$, f is a function of A *onto* B, or $f : A \to B$ is an *onto* (*surjective*) *function.* The function $f : A \to B$ is *one-one* (*injective*) if for any two elements a_1 and a_2 of A, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. A function which is both injective and surjective is called *bijective.*

Let A be any set. Then $f: A \to B$ defined by $f(x) = x$, $x \in A$ is called the *identity function*, denoted by I_A. An identity function is bijective. A function f is called a *constant function* if its range consists of only one element. Let $f : A \to B$ and $g : B \to C$ be two functions such that $f(a) = b$, $a \in A$ and $b \in B$; and $g(b) = c$, where $c \in C$. Then, the function $h : A \to C$ defined by

$$h(a) = c = g(b) = g(f(a)),\ a \in A$$

is called the *composite function* of two functions f and g, denoted by $g \circ f$. If f : $A \to B$, then $I_B \circ f = f$ and $f \circ I_A = f$. Let $f : A \to B$ be a function and $E \subset A$. The function $f \circ I_E : E \to B$ is called the *restriction* of f to the set E, denoted by f/E; dually, the function f is referred to as the *extension* of f/E to the set A.

Let $f : A \to B$ and $b \in B$. Then the *f-inverse* of b, denoted by $f^{-1}(b)$, consists of those elements of A which are mapped onto b by f, i.e. those elements of A which have b as their f-image. More briefly, if $f : A \to B$, then

$$f^{-1}(b) = \{x \in A : f(x) = b\}$$

It is obvious that $f^{-1}(b)$ is a subset of A. We read f^{-1} as 'f-inverse'. It is easy to verify that a function $f : A \to B$ is injective if and only if for each $b \in B$, $f^{-1}(b)$ is either empty or singleton (set consisting of only one element). Let $f : A \to B$ and

B_1 be a subset of B. Then, the inverse image of B_1 under the function f, denoted by $f^{-1}(B_1)$, consists of those elements of A which are mapped by f onto an element in B_1. More briefly,

$$f^{-1}(B_1) = \{x \in A : f(x) \in B_1)$$

It is easy to prove that a function $f : A \to B$ is onto if and only if for every non-empty subset $B_1 \subset B$, $f^{-1}(B_1)$ is a non-empty set. For a function $f : A \to B$ which is bijectice, we note that

$$f^{-1} \circ f = I_A \quad \text{and} \quad f \circ f^{-1} = I_{B.}$$

It may also be seen that $\iota_A^{-1} = I_B$.

Let X and Y be two non-empty sets and $f: X \to Y$ be a function. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of X and $\{B_\alpha\}_{\alpha \in \Lambda}$ be a family of subsets of Y. Then:

(i) $A_\alpha \subset A_\beta \Rightarrow f(A_\alpha) \subset f(A_\beta)$, $\alpha, \beta \in \Lambda$
(ii) $B_\alpha \subset B_\beta \Rightarrow f^{-1}(B_\alpha) \subset f^{-1}(B_\beta)$, $\alpha, \beta \in \Lambda$
(iii) $f(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} f(A_\alpha)$
(iv) $f(\bigcap_{\alpha \in \Lambda} A_\alpha) \subset \bigcap_{\alpha \in \Lambda} f(A_\alpha)$
(v) $f^{-1}(\bigcup_{\alpha \in \Lambda} B_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$
(vi) $f^{-1}(\bigcap_{\alpha \in \Lambda} B_\alpha) = \bigcap_{\alpha \in \Lambda} f^{-1}(B_\alpha)$
(vii) $A_\alpha \subset f^{-1}(f(A_\alpha))$
(viii) $f(f^{-1}(B_\alpha)) \subset B_\alpha$
(ix) $f^{-1}(f(A_\alpha)) = A_\alpha$ if f is injective
(x) $f(f^{-1}(B_\alpha)) = B_\alpha$ if f is subjective
(xi) $f^{-1}(Y - B_\alpha) = X - f^{-1}(B_\alpha)$

1.3 RELATIONS

Let S be any set. A binary *relation* $\mathbf{R}$ on S is defined as subset of $S \times S$. If $\mathbf{R}$ is a relation on a set S, then for $x, y \in S$ we write $x\mathbf{R}y$ to mean $(x, y) \in \mathbf{R}$ and read it as 'x is related to y under $\mathbf{R}$'. A relation $\mathbf{R}$ defined on a set S is said to be *reflexive* if $x\mathbf{R}x$ for every $x \in S$; *symmetric* if $x\mathbf{R}y$ implies $y\mathbf{R}x$; and *transitive* if $x\mathbf{R}y$ and $y\mathbf{R}z$ imply $x\mathbf{R}z$. A relation is said to be an *equivalence relation* if it is reflexive, symmetric and transitive. Another concept closely associated to the equivalence relation is that of *partition* of a set. A *partition* of a set S is a pairwise disjoint collection of non-empty subsets of S whose union is S. *An equivalence relation in S defines a partition of S and, conversely, a partition of S yields on equivalence relation in S.* Let $\mathbf{R}$ be an equivalence relation in S. Then for each $s \in S$, let

$$\mathbf{R}(s) = \{x \in S : x\mathbf{R}s)$$