CONTEMPORARY MATHEMATICS

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Proceedings of the Brezis-Browder Conference
Noncompact Variational Problems
and General Relativity
October 14–18, 2001
Rutgers, The State University of New Jersey,
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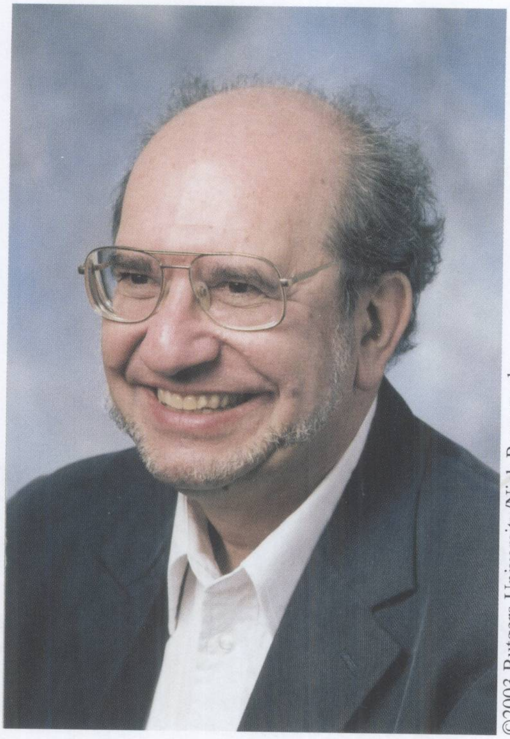
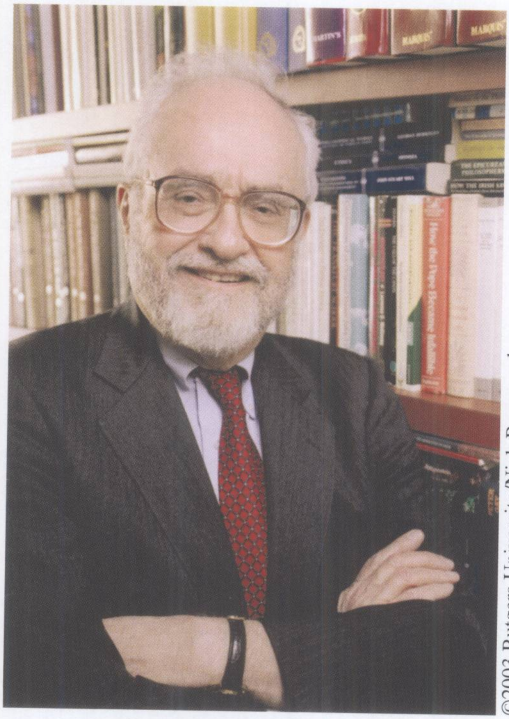


Haïm Brezis



Felix Browder

Felix Browder and Haïm Brezis

Preface

It is a great pleasure and an honour to write this introduction to the Proceedings of the Brezis-Browder Conference which was held at Rutgers, The State University of New Jersey from October 14th to October 18th 2001.

The Conference had two purposes: it was conceived and meant to honour two great living mathematicians, Haim Brezis and Felix Browder who have had and continue to have, each in his own way and through their intense collaboration, a profound impact on the fields of Partial Differential Equations, Functional Analysis, and Geometry.

This conference was also conceived as a gathering (with appropriate timing and momentum) of mathematicians with interests in non compact variational problems, pseudo-holomorphic curves, singular and smooth solutions to problems admitting a conformal (or some group) invariance, Sobolev spaces on manifolds, and Configuration spaces. In addition S. Klainerman organized a day around Einstein equations and related topics.

The speakers and participants came from all around the world: the U.S., France, Italy, Germany, China, Israel, and from the Third World (Tunisia, Mauritania).

I would like to point out that Haim Brezis and Felix Browder, besides their outstanding contribution to mathematics, have also contributed in an exceptional way to the thriving of the mathematical community. The students of H. Brezis have multiplied throughout countries and continents, in an almost biblical way (for a mathematician). The contribution of Felix Browder to the mathematical community is underlined by the number of conferences and symposia, seminars etc. which he has organized and also by his role as president of the AMS.

Both of them also have the very special distinction that they have engaged, directly and indirectly, in quite successful efforts to promote mathematics in areas of the world where it used to have little impact (e.g. North and Black Africa).

The Conference, with respect to its original design, was a success.

However, it was overshadowed by the tragic events of September 11th 2001, which filled the hearts of all speakers, participants, organizers with a deep sense of sadness and delusion.

After September 11th, all speakers and participants thought of this conference also in a new way, as a collective act of protest, of friendship, of knowledge. It was rededicated to the memory of the victims of September 11th, 2001.

One year and a half later, I think that we have achieved our purpose, which in the meantime had become at least threefold: the one which we have chosen early on, that is to honour Haim and Felix and to discuss noncompact phenomena, Einstein equations etc; the one which we came to choose: to honour the memory of the victims of September 11th, and then also the simpler one of going on with continuous questioning in Science, in a friendly and positive atmosphere, through our talks, papers, seminars, and conferences.

For the organizing Committee,
Abbas Bahri

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Conformal Deformations of Riemannian Metrics via "Critical Point Theory at Infinity" : the Conformally Flat Case with Umbilic Boundary

Mohameden Ould Ahmedou

Dedicated to Haim Brezis and Felix Browder, and to the memory of the victims of **9/11**

ABSTRACT. In this paper we prove that every Riemannian metric on a locally conformally flat manifold with umbilic boundary can be conformally deformed to a scalar flat metric having constant mean curvature. This result can be seen as a generalization to higher dimensions of the well known Riemann mapping Theorem in the plane.

1. Introduction

In [**16**], José F. Escobar raised the following question: Given a compact Riemannian manifold with boundary, when it is conformally equivalent to one that has zero scalar curvature and whose boundary has a constant mean curvature ? This problem can be seen as a "generalization" to higher dimensions of the well known Riemannian mapping Theorem. The later states that an open, simply connected proper subset of the plane is conformally diffeomorphic to the disk. In higher dimensions few regions are conformally diffeomorphic to the ball. However one can still ask whether a domain is conformal to a manifold that resembles the ball into ways : namely, it has zero scalar curvature and its boundary has constant mean curvature. In the above the term "generalization" has to be understood in that sens. The above problem is equivalent to finding a smooth positive solution to the following nonlinear boundary value problem on a Riemannian manifold with boundary (M^n, g), $n \geq 3$:

$$(P) \qquad \begin{cases} -\Delta_g u + \frac{(n-2)}{4(n-1)} R_g u = 0, \quad u > 0 & \text{in } \overset{\circ}{M}; \\ \partial_\nu u + \frac{n-2}{2} h_g u = Q(M, \partial M) u^{\frac{n}{n-2}}, & \text{on } \partial M. \end{cases}$$

where R is the scalar curvature of M, h is the mean curvature of ∂M, ν is the outer normal vector with respect to g and $Q(M, \partial M)$ is a constant whose sign is uniquely determined by the conformal structure. Indeed if $\overline{g} = u^{\frac{4}{n-2}} g$, then the metric $\overline{g}$ has

Key words and phrases. Critical trace Sobolev exponent, curvature, conformal invariance, lack of compactness, critical point at infinity .

zero scalar curvature and the boundary has constant mean curvature with respect to $\overline{g}$.

Solutions of equation (P) correspond , up to a multiple constant, to critical points of the following functional J defined on $H^1(M) \setminus \{0\}$

$$(1) \qquad J(u) = \frac{\left(\int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2\right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g\right)^{\frac{n-1}{n-2}}}{\int_{\partial M} |u|^{2\frac{n-1}{n-2}} d\sigma_g}.$$

where dV_g and $d\sigma_g$ denote the Riemannian measure on M and ∂M induced by the metric g.

The regularity of the H^1 solutions of (P) was established by P. Cherrier [**13**]; and related problems regarding conformal deformations of metrics on manifold with boundary were studied in [**1**] ,[**8**] [**11**] , [**14**], [**19**] , [**20**][**21**], [**22**] , [**23**], [**26**] , [**29**] and the references therein.

The exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(M) \to L^q(\partial M)$. This embedding being not compact , the functional J does not satisfy the Palais Smale condition. For this reason standard variational methods cannot be applied to find critical points of J.

Following the original arguments introduced by T. Aubin [**2**], [**3**] and R. Schoen [**31**] to prove Yamabe conjecture on closed manifolds, Escobar proved the existence of a smooth positive solution u of (P) on $(M^n, g), n \geq 3$ for many cases. To state his results we need some preliminaries:

Let H denote the second fondamental form of ∂M in (M, g) with respect to the inner normal. Let us denote the traceless part of the second fundamental form by U that is $U(X,Y) = H(X,Y) - h_g\, g(X,Y)$

Definition 1.1. *A point $q \in \partial M$ is called an* **umbilic point** *if $U = 0$ at q. ∂M is called umbilic if every point of ∂M is umbilic.*

Regarding the above problem Escobar proved the following Theorem [**16, 18**]:

Theorem 1.1. *Let (M^n, g) be a compact Riemannian manifold with boundary, $n \geq 3$. Assume that M^n satisfies one of the following conditions:*

: *(i) $n \geq 6$ and M has a nonumbilic point on ∂M*
: *(ii) $n \geq 6$ and M is conformally locally flat with umbilic boundary*
: *(iii) $n = 4, 5$ and ∂M is umbilic*
: *(vi) $n = 3$*

then there exists a smooth metric $u^{\frac{4}{n-2}} g$, $u > 0$ on M of zero scalar curvature and constant mean curvature on ∂M.

In his proof Escobar uses strongly an extension of the positive mass Theorem of R. Schoen and S.T. Yau [**33**], [**32**] to some type of manifolds with boundary. Such an extension was proved by Escobar in [**17**]. Besides the proof of T.Aubin and R.Schoen of the Yamabe conjecture, another proof by A. Bahri [**5**] and A.Bahri and H. Brezis [**6**] of the same conjecture is available by techniques related to the Theory of critical point at Infinity of A. Bahri [**4**].

We plan to give a complete positive answer to the above problem based on the topological argument of Bahri-Coron [**7**], as Bahri and Brezis did for the Yamabe conjecture. In this first part we study the case where the manifold is locally conformally flat with umbilic boundary. Namely we prove the following Theorem

THEOREM 1.2. *Suppose that* (M^n, g), $n \geq 3$ *is a compact locally conformally flat manifold with umbilic boundary, then equation (P) has a solution.*

Let us observe that while the solution obtained by Escobar is a minimum of J, our solution is in general, a critical point of J of higher Morse index, more precisely we have the following characterization of solutions obtained by Bahri-Coron existence scheme(see **[12]**) :

THEOREM 1.3. *The solution* u *obtained in Theorem 1.2 satisfies, for some integer* p_0*:*

: *(i)* $p_0^{\frac{1}{n-2}} S \leq J(u) \leq (p_0+1)^{\frac{1}{n-2}} S$

: *(ii)* $ind(J,u) \leq (p_0+1)(n-1)+p_0$, $ind(J,u) + dimkerd^2 J(u) \geq p_0(n-1)+p_0$

: *(iii)* u *induces some difference of topology at the level* $p_0^{\frac{1}{n-2}} S$.

where $ind(J,u)$ *is the Morse index of* J *at* u, *and* $S = 2^{1-n}\omega_{n-1}$, *where* ω_{n-1} *is the volume of the* $n-1$ *dimensional unit sphere. Moreover, if (P) has only nondegenerate solutions,* $ind(J,u) = (n-1)p_0 + p_0$.

The remainder of the paper is organized as follows: in section 2 we construct some "almost solutions" which are solutions of the "problem at Infinity". In section 3 we collect some standard results regarding the description of the lack of compactness and some local deformation Lemma. In section 4 we perform an expansion of J at 'Infinity' and we give the proof of Theorem 1.2 in section 5. Lastly we devote the appendix to establish some technical Lemma and to recall some well known results.

Acknowledgements

This paper is dedicated to Prof. H. Brezis and F. Browder and to the memory of the Victims of **9/11**. The author is indebted to Pr. Abbas Bahri for teaching him his Theory of critical point at Infinity and he is gratefull to Pr. Antonio Ambrosetti for his interest in his work and his constant support.

2. Construction of "almost solutions"

In this paper we assume that (M^n, g) is a compact Riemannian manifold with boundary and dimension $n \geq 3$. Let R_{pq} and $R = g^{pq}R_{pq}$ be the Ricci curvature and the scalar curvature, respectively; let h_{ij} and $h = \frac{1}{n-1} g^{ij} h_{ij}$ be the second fundamental form of the boundary of M, ∂M and the mean curvature , respectively. Let $\tilde{g} = u^{\frac{4}{n-2}} g$ be a metric conformally related to g. We denote by a tilde all quantities computed with respect to the metric $\tilde{g}$. The transformation law for the scalar curvature is

$$\tilde{R} = \frac{4(n-1)}{n-2} \frac{Lu}{u^{\frac{n+2}{n-2}}} \tag{2}$$

where L is the conformal Laplacian $L = \Delta - \frac{n-2}{4(n-1)} R$ on M; while the transformation law for the mean curvature is

$$\tilde{h} = \frac{2}{n-2} \frac{Bu}{u^{\frac{n}{n-2}}} \tag{3}$$

where B is the boundary operator $B = \frac{\partial}{\partial \nu} + \frac{n-2}{2} h$ on ∂M.

Consider now the following eigenvalue problem on (M, g) :

$$(E) \qquad \begin{cases} L_g u = \lambda u & \text{on } \overset{\circ}{M} \\ B_g u = 0 & \text{on } \partial M \end{cases}$$

Let λ_1 the first eigenvalue of (E).

DEFINITION 2.1. *We say that a manifold M is of positive(negative, zero) type if $\lambda_1 > 0 (< 0, = 0)$.*

As it is well known the existence problem is easy when the manifold is of negative or zero type, so we treat only in this paper the case of manifold of positive type.
Now we construct some almost solutions of (P), which will play a central role in the description of the lack of compactness.

Let f_1 denote a positive eigenfunction corresponding to the first eigenvalue of (E) , and consider $g_1 = (f_1)^{\frac{4}{n-2}} g$ then , according to (2) and (3) we have: $R_{g_1} > 0$ and $h_{g_1} = 0$ on ∂M. We can work with g_1 instead of g, but for simplicity we still denote it by g. Let $a \in \partial M$; since M is a compact locally conformally flat manifold one can find a neighborhood of a, $\mathcal{U}(a) \supset B_\rho^M(a)$, $\rho > 0$ uniform and a conformal diffeomorphism φ which maps $B_\rho^M(a)$ into $\mathbb{R}^n$ with $\varphi(0) = a$. Therefore, denoting g_0 the flat metric on $\mathbb{R}^n$, there exist a positive function u_a such that $\varphi^*(g_0) = u_a^{\frac{4}{n-2}} g$. Since the boundary is umbilic, $\varphi(\partial M \cap B_\rho^M(a))$ has to be a piece of sphere or a piece of a hyperplane (See [**34**]) and since spheres and hyperplanes are locally conformal to each other, we can assume without loss of generality that $\partial B_2^+(0) \cap \partial \mathbb{R}_+^n \subset \varphi(\partial M \cap B_\rho^M(a))$ and $\varphi(\overset{\circ}{M} \cap B_\rho^M(a)) \subset \mathbb{R}_+^n$. Since $\partial B_2^+(0) \cap \partial \mathbb{R}_+^n$ has zero mean curvature in $\overline{B_2^+}$, we deduce from (3) that $\frac{\partial u_a}{\partial \nu} = 0$ on $\partial M \cap B_\rho^M(a)$. We extend u_a to be a smooth positive function on M such that $\frac{\partial u_a}{\partial \nu} = 0$ on ∂M and $u_a = 0$ on $M \setminus B_{2\rho}^M(a)$. Consider now the conformal metric $\overline{g_0} = u^{\frac{4}{n-2}} g$, then $\overline{g_0}$ has the property that $h_{\overline{g_0}} = 0$ and it is Euclidean in $B_\rho^M(a)$. Moreover this metric can be chosen to depend smoothly on a (see [**5**]).

For $a \in \partial M$, define the function:

$$\delta_{a,\lambda}(y) = \overline{c} \frac{\lambda^{\frac{n-2}{2}}}{((1 + \lambda x^n)^2 + \lambda^2 |x'|^2)^{\frac{n-2}{2}}}$$

where $(x', x^n) = \varphi(y)$, and $\overline{c}$ is chosen such that $\delta_{a,\lambda}$ satisfies the following equation

$$\begin{cases} -\Delta_{\overline{g}_0} u = 0, & \text{in } B_\rho^M \cap \overset{\circ}{M}; \\ \partial_\nu u = \delta_{a,\lambda}^{\frac{n}{n-2}}, & \text{on } B_\rho^M \cap \partial M \end{cases}$$

Set $\hat{\delta}_{a,\lambda} = \omega_a u_a \delta_{a,\lambda}$ where ω_a is a cutoff function $\omega_a = 1$ on $B_\rho^M(a)$ and $\omega_a = 0$ on $M \setminus B_{2\rho}^M$.

We define now a familly of almost solutions $\varphi_{a,\lambda}$ to be the unique solution of

$$\begin{cases} -L_g u = 0, & \text{in } \overset{\circ}{M}; \\ B_g u = \hat{\delta}_{a,\lambda}^{\frac{n}{n-2}}, & \text{on } \partial M \end{cases}$$

Let us recall that the operators L_g and B_g are conformally invariant under the conformal change of metrics, namely we have:

LEMMA 2.1. **[16]**
Let $\psi \in C^2(B_\rho(a))$, we have

$$L_g(u_a\psi) = u_a^{\frac{n+2}{n-2}} L_{\overline{g}_0}(\psi)$$

and

$$B_g(u_a\psi) = u_a^{\frac{n}{n-2}} B_{\overline{g}_0}(\psi$$

In the remainder of this section we establish some properties of our almost solutions $\varphi_{a,\lambda}$.

LEMMA 2.2. *There are two positive constants C and B , such that for all $a \in \partial M$ and $\lambda \geq B$, we have*

$$\left|\varphi_{a,\lambda} - \hat{\delta}_{a,\lambda}\right|_\infty \leq \frac{C}{\lambda^{\frac{n-2}{2}}}$$

PROOF.
Let $H_{a,\lambda} = \lambda^{\frac{n-2}{2}}(\varphi_{a,\lambda} - \hat{\delta}_{a,\lambda})$, we have

$$L_g\, H_{a,\lambda} = \lambda^{\frac{n-2}{2}} \quad L_g\,(\omega_a u_a \delta_{a,\lambda}) = \lambda^{\frac{n-2}{2}}\, u_a^{\frac{n+2}{n-2}} \quad L_g\,(\omega_a \delta_{a,\lambda})$$

Since on B_ρ , $\omega_a = 0$, we deduce that on B_ρ we have $L_g\, H_{a,\lambda} = 0$, whereas on $M \setminus B_\rho$ there holds $L_g\, H_{a,\lambda} \leq C$.
From another part

$$B_g\, H_{a,\lambda} = \lambda^{\frac{n-2}{2}} \quad [B_g\, \varphi_{a,\lambda} - B_g\,(\omega_a u_a \delta_{a,\lambda})] = \lambda^{\frac{n-2}{2}}\, [\hat{\delta}_{a,\lambda} \quad - u_a^{\frac{n}{n-2}} \quad B_g\,(\omega_a \delta_{a,\lambda})]$$

on $B_\rho(a) \cap \partial M$, $\omega_a = 1$, therefore $B_g\, H_{a,\lambda} = 0$, while on $M \setminus B_\rho$ there holds $B_g\, H_{a,\lambda} \leq C$. Thus our Lemma follows from Lemma 6.3 quoted in the appendix.
■

LEMMA 2.3. *There are two positive constants C and B , such that for all $a \in \partial M$ and $\lambda \geq B$, we have*

$$\varphi_{a,\lambda} \geq \frac{C}{\lambda^{\frac{n-2}{2}}}$$

PROOF.
Using Lemma 2.2, we know that if $\rho_1 < \rho$ is chosen small enough, independent of λ, the following inequality holds on $B(a, \rho_1)$

$$\varphi_{a,\lambda} \geq \hat{\delta}_{a,\lambda} - \frac{C}{\lambda^{\frac{n-2}{2}}} \geq \frac{C}{\lambda^{\frac{n-2}{2}}}$$

Let $\Sigma_1 = \partial B(a,\rho) \cap \mathring{M}$ and $\Sigma_2 = \partial M \setminus \Sigma_1$.
Then we have

$$\begin{cases} L_g(\varphi_{a,\lambda} - \frac{C}{\lambda^{\frac{n-2}{2}}}) & \leq 0 \quad \text{in } \mathring{M} \\ \varphi_{a,\lambda} - \frac{C}{\lambda^{\frac{n-2}{2}}} & \geq 0, \quad \text{on } \Sigma_1 \\ \frac{\partial}{\partial \nu}(\varphi_{a,\lambda} - \frac{C}{\lambda^{\frac{n-2}{2}}}) & \geq 0 \quad \text{on } \Sigma_2 \end{cases} \tag{4}$$

Then by the hopf maximum principle, we deduce from (4) that

$$\varphi_{a,\lambda} \geq \frac{C}{\lambda^{\frac{n-2}{2}}} \quad \text{for } x \in M$$

■

LEMMA 2.4. *Let $\theta > 0$ be given. There are positive constants C and B, such that the following estimates hold, provided $\lambda \geq B$*

: *(i)*

$$\left|\int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g \quad - \overline{c}^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+|x|^2)^{n-1}}\right| \quad \leq \frac{C}{\lambda^{n-2}} \quad \textit{for } a \in \partial M$$

: *(ii)*

$$\left|\int_{\partial M} \varphi_{a,\lambda}^{\frac{2(n-1)}{n-2}} d\sigma_g \quad - \overline{c}^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+|x|^2)^{n-1}}\right| \quad \leq \frac{C}{\lambda^{n-2}} \quad \textit{for } a \in \partial M$$

: *(iii)*

$$\int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}} \varphi_{a_2,\lambda} d\sigma_g \quad \geq \frac{C}{\lambda^{n-2}} \quad \textit{for } a_1, a_2 \in \partial M$$

: *(vi)*

$$\int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g \quad \leq (1+\theta) \int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}} \varphi_{a_2,\lambda} d\sigma_g$$

PROOF.

Proof of (i)

From the definition of $\varphi_{a,\lambda}$, we derive:

$$\int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g = \int_{\partial M} (\omega_a \delta_{a,\lambda} u_a)^{\frac{n}{n-2}} \varphi_{a,\lambda} d\sigma_g$$

Using Lemma 2.2, we deduce:

$$\begin{aligned}
(5) \quad & \int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g \\
= & \int_{\partial M \cap B_{2\rho}} (\omega_a \delta_{a,\lambda} u_a)^{\frac{2(n-1)}{n-2}} d\sigma_g + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M \cap B_{2\rho}} (\omega_a \delta_{a,\lambda} u_a)^{\frac{n}{n-2}} d\sigma_g \\
= & \int_{\partial M \cap B_{\rho}} (\delta_{a,\lambda})^{\frac{2(n-1)}{n-2}} dv + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M \cap B_{2\rho}} (\omega_a \delta_{a,\lambda} u_a)^{\frac{n}{n-2}} dv_{g_0} + O(\lambda^{n-1}) \\
= & \overline{c}^{\frac{2(n-1)}{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+|x|^2)^{n-1}} + O(\frac{1}{\lambda^{n-2}}).
\end{aligned}$$

The proof of (ii) is essentially reduced, up to minor differences to the same computations involved in the proof of (ii).

Proof of (iii)

From Lemma 2.3 we deduce

$$\int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}} \varphi_{a_2,\lambda} d\sigma_g \geq \frac{1}{C\lambda^{n-2}} \int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}} d\sigma_g$$

Then from Lemma 2.2 and Lemma 2.3 we derive

$$\begin{aligned}
\int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}} \varphi_{a_2,\lambda} d\sigma_g \quad & \geq \frac{1}{C\lambda^{n-2}} \int_{B_\rho(a) \cap \partial M} \hat{\delta}_{a_1,\lambda}^{\frac{n}{n-2}} d\sigma_g \\
& \geq \frac{1}{C\lambda^{n-2}} \int_{B_\rho(a) \cap \partial M} \delta_{a_1,\lambda}^{\frac{n}{n-2}} d\sigma_{g_0} = O(\frac{1}{\lambda^{n-2}})
\end{aligned}$$

Proof of (vi)

$$\begin{aligned}
&\int_{\partial M} B_g\varphi_{a_1,\lambda}\,\varphi_{a_2,\lambda}dv_g \\
= &\int_{\partial M} \hat\delta_{a_1,\lambda}^{\frac{n}{n-2}}\,\varphi_{a_2,\lambda}dv_g \\
= &\int_{\partial M\cap B_\rho} \varphi_{a_1,\lambda}^{\frac{n}{n-2}}\varphi_{a_2,\lambda} + O(\frac{1}{\lambda^{\frac{n}{2}}})\int_{\partial M\setminus B_\rho}\varphi_{a_2,\lambda} \\
= &\int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}}\,\varphi_{a_2,\lambda}d\sigma_g + O(\frac{1}{\lambda^{\frac{n-2}{2}}})\int_{\partial M}\hat\delta_{a_1,\lambda}^{\frac{2}{n-2}}\,\varphi_{a_2,\lambda}d\sigma_g + O(\frac{1}{\lambda^{n-1}}) \\
= &\int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}}\,\varphi_{a_2,\lambda}dv_g\,\varphi_{a_2,\lambda} + O(\lambda^{\frac{n-2}{2}})\int_{\partial M}\delta_{a_1,\lambda}^{\frac{2}{n-2}}\,\varphi_{a_2,\lambda}d\sigma_{\overline{g}_0} \\
= &\int_{\partial M} \varphi_{a_1,\lambda}^{\frac{n}{n-2}}\,\varphi_{a_2,\lambda}dv_g + O(\frac{1}{\lambda^{\frac{n-2}{2}}})\int_{\partial M}\delta_{a_1,\lambda}^{\frac{2}{n-2}}\,\delta_{a_2,\lambda}d\sigma_{\overline{g}_0} + O(\frac{1}{\lambda^{n-1}})
\end{aligned}$$

Now from Lemma 6.1 in the Appendix we deduce :

$$O(\frac{1}{\lambda^{\frac{n-2}{2}}})\int_{\partial M\cap B_\rho(a_1)}\delta_{a_1,\lambda}^{\frac{n}{n-2}}\,\delta_{a_2,\lambda}d\sigma_{\overline{g}_0} \quad = \quad o\left(\int_{\partial M}\delta_{a_1,\lambda}^{\frac{n}{n-2}}\,\delta_{a_2,\lambda}d\sigma_{\overline{g}_0}\right)$$

Therefore using (iii) we have

$$\int_{\partial M} B_g\varphi_{a_1,\lambda}\,\varphi_{a_2,\lambda}dv_g = \int_{\partial M}\varphi_{a_1,\lambda}^{\frac{n}{n-2}}\,\varphi_{a_2,\lambda}dv_g(1+o(1))$$

The proof of (vi) and the proof of Lemma 2.4 are thereby completed. ■

3. Some standard facts

We recall that solutions of Problem (P) arises , up to a constant, as critical points of the functional J is defined by

$$J(u) = \left(\int_M -L_g u\,u\,dv_g + \int_{\partial M} B_g u\,u\,d\sigma_g\right)^{\frac{n-1}{n-2}}\left(\int_{\partial M} u^{\frac{2(n-1)}{n-2}}\right)^{-1}$$

where u belongs to Σ^+ defined as follows:

$$\Sigma^+ = \{u \in H^1(M), u \geq 0, \|u\| = 1\}$$

Let us observe that Σ^+ is invariant by the flow of $-\partial J$.

The functional J is known to not satisfy Palais Smale condition(PS for short) , which leads to the failure of classical existence mecanism. In order to describe this failure we need some notation.

For $\varepsilon > 0$ and $p \geq 1$, let

$$V(p,\varepsilon) = \left\{\begin{array}{l} u \in \Sigma^+ \text{such that } \exists(a_1,\cdots,a_p) \in (\partial M)^p \text{ and } \exists\,(\lambda_1,\cdots,\lambda_p) \in (\mathbb{R}_+^*)^p \text{such that} \\ \qquad \left\|u - \frac{\sum_{i=1}^p \varphi_{a_i,\lambda_i}}{\|\sum_{i=1}^p \varphi_{a_i,\lambda_i}\|}\right\| < \varepsilon, \text{ with } \lambda_i \geq \frac{1}{\varepsilon} \text{ and } \varepsilon_{ij} < \varepsilon \end{array}\right\}$$

where $\varepsilon_{ij} = \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\, d(a_i,a_j)^2}\right)^{\frac{n-2}{2}}$ and d denotes the geosedic distance.