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Representation Theory and Higher Algebraic K-Theory

Aderemi Kuku

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Introduction

A representation of a discrete group G in the category $\mathcal{P}(F)$ of finite dimensional vector spaces over a field F could be defined as a pair $(V, \rho : G \to Aut(V))$ where $V \in \mathcal{P}(F)$ and ρ is a group homomorphism from G to the group $Aut(V)$ of bijective linear operators on V. This definition makes sense if we replace $\mathcal{P}(F)$ by more general linear structures like $\mathcal{P}(R)$, the category of finitely generated projective modules over any ring R with identity.

More generally, one could define a representation of G in an arbitrary category $\mathcal{C}$ as a pair $(X, \rho : G \to Aut(X))$ where $X \in ob(\mathcal{C})$ and ρ is a group homomorphism from G to the group of $\mathcal{C}$-automorphisms of X. The representations of G in $\mathcal{C}$ also form a category $\mathcal{C}_G$ which can be identified with the category $[\underline{G/G}, \mathcal{C}]$ of covariant functors from the translation category $\underline{G/G}$ of the G-set G/G (where G/G is the final object in the category of G-sets (see 1.1). The foregoing considerations also apply if G is a topological group and $\mathcal{C}$ is a topological category, i.e. a category whose objects X and $Hom_{\mathcal{C}}(X,Y)$ are endowed with a topology such that the morphisms are continuous. Here, we have an additional requirement that $\rho : G \to Aut(X)$ be continuous. For example, G could be a Lie group and $\mathcal{C}$ the category of Hilbert spaces over $\mathbb{C}$, in which case we have unitary representations of G.

It is the aim of this book to explore connections between $\mathcal{C}_G$ and higher algebraic K-theory of $\mathcal{C}$ for suitable categories (e.g. exact, symmetric monoidal and Waldhausen categories) when G could be a finite, discrete, profinite or compact Lie group.

When $\mathcal{C} = \mathcal{P}(\mathbb{C})$, ($\mathbb{C}$ the field of complex numbers) and G is a finite or compact Lie group, the Grothendieck group $K_0(\mathcal{C}_G)$ can be identified with the group of generalized characters of G and thus provides the initial contact between representation theory and K-theory. If F is an arbitrary field, G a finite group, $\mathcal{P}(F)_G$ can be identified with the category $\mathcal{M}(FG)$ of finitely generated FG-modules and so, $K_0(\mathcal{P}(F)_G) \cong K_0(\mathcal{M}(FG)) \cong G_0(FG)$ yields K-theory of the group algebra FG, thus providing initial contact between K-theory of $\mathcal{P}(F)_G$ and K-theory of group algebras (see 1.2). This situation extends to higher dimensional K-theoretic groups i.e. for all $n \geq 0$ we have $K_n(\mathcal{P}(F)_G) \cong K_n(\mathcal{M}(FG)) \cong G_n(FG)$ (see 5.2).

More generally, if R is any commutative ring with identity and G is a finite group, then the category $\mathcal{P}(R)_G$ can be identified with the category $\mathcal{P}_R(RG)$ of RG-lattices (i.e. RG-modules that are finitely generated and projective over R) and so, for all $n \geq 0$, $K_n(\mathcal{P}(R)_G)$ can be identified with $K_n(\mathcal{P}_R(RG))$ which, when R is regular, coincides with $K_n(\mathcal{M}(RG))$ usually denoted by $G_n(RG)$ (see $(5.2)^B$).

When R is the ring of integers in a number field or p-adic field F or more generally R a Dedekind domain with quotient field F or more generally still R a regular ring, the notion of a groupring $RG(G$ finite) generalizes to the notion of R-orders Λ in a semi-simple F-algebra Σ when char(F) does not divide the order of G and so, studying K-theory of the category $\mathcal{P}_R(\Lambda)$ of Λ-

lattices automatically yields results on the computations of K-theory of the category $\mathcal{P}_R(RG)$ of RG-lattices and so, K-theory of orders is appropriately classified as belonging to Integral representation theory.

Now the classical K-theory (K_0, K_1, K_2, K_{-n}) of orders and grouprings (especially K_0 and K_1) have been well studied via classical methods and documented in several books [20, 39, 159, 168, 211, 213] and so, we only carefully review the classical situation in Part I of this book (chapters 1-4), with clear definitions, examples, statements of important results (mostly without proofs) and refer the reader to one of the books or other literature for proofs. We include, in particular, classical results which have higher dimensional versions for which we supply proofs once and for all in the context of higher K-theory. Needless to say that some results proved for higher K-theory with no classical analogues invariably apply to the classical cases also. For example, there was no classical result that $K_2(\Lambda), G_2(\Lambda)$, are finite for arbitrary orders Λ in semi-simple algebras over number fields, but we prove in this book that $K_{2n}(\Lambda), G_{2n}(\Lambda)$, are finite for all $n \geq 1$, thus making this result also available for $K_2(\Lambda)$.

Some of the impetus for the growth of Algebraic K-theory from the beginning had to do with the fact that the classical K-group of grouprings housed interesting topological/geometric invariants, e.g.

(1) Class groups of orders and grouprings (which also constitute natural generalizations to number theoretic class groups of integers in number fields) also house Swan-Wall invariants (see $(2.3)^C$ and [214, 216]) etc.

(2) Computations of the groups $G_0(RG)$, R Noetherian, G Abelian is connected with the calculations of the group 'SSF' (see $(2.4)^B$ or [19]) which houses obstructions constructed by Shub and Francs in their study of Morse-Smale diffeomorphisms (see [19]).

(3) Whitehead groups of integral grouprings house Whitehead torsion which is also useful in the classification of manifolds (see [153, 195]).

(4) If G is a finite group and $Orb(G)$ the orbit category of G (an 'EI' category (see 7.6)). X a G-CW-complex with round structure (see [137]), then the equivariant Riedemester torsion takes values in $Wh(Q\ orb(G))$ where $Wh(Q\ orb(G))$ is the quotient of $K_1(QOrb(G))$ by subgroups of "trivial units" see [137].

(5) K_2 of integral grouprings helps in the understanding of the pseudo-isotopy of manifolds (see [80]).

(6) The negative K-theory of grouprings can also be interpreted in terms of bounded h-corbordisms (see $(4.5)^E$ or [138]).

It is also noteworthy that several far-reaching generalizations of classical concepts have been done via higher K-theory. For example, the K-theoretic definition of higher dimensional class groups $C\ell_n(\Lambda)(n \geq 0)$ of orders Λ generalize to higher dimensions the notion of class group $C\ell(\Lambda)$ of orders and grouprings which in turn generalizes the number-theoretic notion of class groups of Dedekind domains and integers R in number fields (see 7.4). Note that $C\ell_1(\Lambda)$ for $\Lambda = RG$ is intimately connected with Whitehead torsion (see 7.4 or [159]) and as already observed $C\ell(\Lambda) = C\ell_0(\Lambda)$ houses some topological/geometric invariants (see $(2.3)^C$).

Moreover, the profinite higher K-theory for exact categories discussed in chapter 8 is a cohomology theory which generalizes classical profinite topological K-theory (see [199]) as well as K-theory analogues of classical continuous cohomology of schemes rooted in Arithmetic algebraic geometry.

Part II (chapters 5 to 8) is devoted to a systematic exposition of higher algebraic K-theory of orders and grouprings. Again, because the basic higher K-theoretic constructions have already appeared with proofs in several books (e.g. [25, 88, 198]), the presentation in chapters 5 and 6 is restricted to a review of important results (with examples) relevant to our context. Topics reviewed in chapter 5 include the 'plus' construction as well as higher K-theory of exact, symmetric monoidal and Waldhausen categories. We try as much as possible to emphasize the utility value of the usually abstract topological constructions.

In chapter 7, we prove quite a number of results on higher K-theory of orders and grouprings. In $(7.1)^A$ we set the stage for arbitrary orders by first proving several finiteness results for higher K-theory of maximal orders in semi-simple algebras over p-adic fields and number fields as well as higher K-theory of associated division and semi-simple algebras.

In $(7.1)^B$, we prove among other results that if R is the ring of integers in a number field F, Λ as R-order in a semi-simple F-algebra Σ, then for all $n \geq 0$, $K_n(\Lambda), G_n(\Lambda)$ are finitely generated Abelian groups, $SK_n(\Lambda), SG_n(\Lambda), SK_n(\hat{\Lambda}_p)$ and $SG_n(\hat{\Lambda}_p)$ are finite groups (see [108, 110, 112, 113]) and $SG_n(RG)$ are trivial (see [131]) where G is a finite group. In 7.2 we prove that rank $K_n(\Lambda)$ = rank $G_n(\Lambda)$ = rank $K_n(\Gamma)$ if Γ is a maximal R-order containing Λ, see [115]. We consequently prove that for all $n \geq 1$, $K_{2n}(\Lambda), G_{2n}(\Lambda)$ are actually finite groups. Hence for any finite group G, $K_{2n}(RG), G_{2n}(RG)$ are finite (see $(7.2)^B$ or [121]).

Next, we obtain in 7.3 a decomposition (for G Abelian) $G_n(RG) \cong \oplus G_n(R < C >)$ for all $n \geq 0$ where R is a Noetherian ring, and C ranges over all cyclic quotients of G and $R < C >= R\zeta_{|C|}(\frac{1}{|C|}), \zeta_{|C|}$ being a primitive $|C|^{\text{th}}$ root of unity (see [232]). (This decomposition is a higher dimensional version of that of $G_0(RG)$ (see $(2.4)^A$.) The decomposition of $G_n(RG)$ is extended to some non-Abelian groups e.g. dihedral, quaternion and nilpotent groups (see [231, 233]). We conclude 7.3 with a discussion of a conjecture due to Hambleton, Taylor and Williams on the decomposition for $G_n(RG)$, G any finite groups (see [76]), and the counter-example provided for this conjecture by D. Webb and D. Yao (see [235].

Next, in 7.4 we define and study higher-dimensional class groups $C\ell_n(\Lambda)$ of R-orders Λ which generalize the classical notion of class groups $C\ell(\Lambda)(= C\ell_0(\Lambda))$ of orders. We prove that $\forall\ n \geq 0$, $C\ell_n(\Lambda)$ is a finite group and identify p-torsion in $C\ell_{2n-1}(\Lambda)$ for arbitrary orders Λ (see [102]) while we identify p-torsion for all $C\ell_{2n}(\Lambda)$ when Λ is an Eichler or hereditary order (see [74, 75]).

In 7.5, we study higher K-theory of grouprings of virtually infinite cyclic groups V in the two cases when $V = G \rtimes_\alpha T$, the semi-direct product of a finite group G (of order r, say) and an infinite cyclic group $T =< t >$ with respect to the automorphism $\alpha : G \to G\ g \to tgt^{-1}$ and when $V = G_0 *_H G_1$ where the groups $G_i = 0, 1$ and H are finite and $[G_i : H] = 2$. These groups V are conjectured by Farrell and Jones (see [54]) to constitute building blocks for the understanding of K-theory of grouprings of an arbitrary discrete group G - hence their importance. We prove that when $V = G \rtimes_\alpha T$, then for all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group and that $NK_n(RV)$ is r-torsion. For $V = G_0 *_H G_1$ we prove that the nil groups of V are $|H|$-torsion (see [123]).

The next section of chapter 7 is devoted to the study of higher K and G-theory of modules over 'EI'-categories. Modules over 'EI'-categories constitute natural generalizations for the notion of modules over grouprings and K-theory of such modules are known to house topological and geometric invariants and are also replete with applications in the theory of transformation groups (see [137]). Here, we obtain several finiteness and other results which are extension of results earlier obtained for higher K-theory of grouprings of finite groups.

In 7.7 we obtain several finiteness results on the higher K-theory of the category of representations of a finite group G in the category of $\mathcal{P}(\Gamma)$ where Γ is a maximal order in central division algebra over number fields and p-adic fields. These results translate into computations of $G_n(\Gamma G)$ as well as lead to showing via topological and representation theoretic techniques that a non-commutative analogue of a fundamental result of R.G. Swan at the zero-dimensional level does not hold (see [110]).

In chapter 8, we define and study profinite higher K and G-theory of exact categories, orders and grouprings. This theory is an extraordinary cohomology theory inspired by continuous cohomology theory in algebraic topology and arithmetic algebraic geometry. The theory yields several ℓ-completeness theorems for profinite K and G-theory of orders and grouprings as well as yields some interesting computations of higher K-theory of p-adic orders otherwise inaccessible. For example we use this theory to show that if Λ is a p-adic order in a p-adic semi-simple algebra Σ, then for all $n \geq 1$, $K_n(\Lambda)_\ell, G_n(\Lambda)_\ell, K_n(\Sigma)_\ell$ are finite groups provided ℓ is a prime $\neq p$. We also define and study continuous K-theory of p-adic orders and obtain a relationship between profinite and continuous K-theory of such orders (see [117]).

Now if $\underline{S}$ is the translation category of any G-set S, and $\mathcal{C}$ is a small category, then the category $[\underline{S}, \mathcal{C}]$ of covariant functors from $\underline{S}$ to $\mathcal{C}$ is also called the category of G-equivariant $\mathcal{C}$-bundles on S because if $\mathcal{C} = \mathcal{P}(\mathbb{C})$, then $[\underline{S}, \mathcal{P}(\mathbb{C})]$ is just the category of G-equivariant $\mathbb{C}$-bundles on the discrete G-space S so that $K_0[\underline{S}, \mathcal{P}(\mathbb{C})] = K_0^G(S, \mathcal{P}(\mathbb{C}))$ is the zero-dimensional G-equivariant K-theory of S. Note that if S is a G-space, then the translation category $\underline{S}$ of S as well as the category $[\underline{S}, \mathcal{C}]$ are defined similarly. Indeed, if S is a compact G-space then $K_0^G(S, \mathcal{P}(\mathbb{C}))$ is exactly the Atyah-Segal equivariant K-theory of S (see [184]).

One of the goals of this book is to exploit representation theoretic techniques (especially induction theory) to define and study equivariant higher algebraic K-theory and their relative generalizations for finite, profinite and compact Lie group actions, as well as equivariant homology theories for discrete group actions in the context of category theory and homological algebra with the aim of providing new insights into classical results as well as open avenues for further applications. We devote Part III (chapters 9 - 14) of this book to this endeavour.

Induction theory has always aimed at computing various invariants of a given group G in terms of corresponding invariants of certain classes of subgroups of G. For example if G is a finite group, it is well know by Artin induction theorem that two G-representations in $\mathcal{P}(\mathbb{C})$ are equivalent if their restrictions to cyclic subgroups of G are isomorphic. In other words, given the exact category $\mathcal{P}(\mathbb{C})$, and a finite group G, we have found a collection $D(\mathcal{P}(\mathbb{C}), G)$ of subgroups (in this case cyclic subgroups) of G such that two G-representations in $\mathcal{P}(\mathbb{C})$ are equivalent iff their restrictions to subgroups in $D(\mathcal{P}(\mathbb{C}), G)$ are equivalent. One could then ask the following general question: Given a category $\mathcal{A}$ and a group G, does there exist a collection $D(\mathcal{A}, G)$ of proper subgroups of G such that two G-representations in $\mathcal{A}$ are equivalent if their restrictions to subgroups in $D(\mathcal{A}, G)$ are equivalent?

As we shall see in this book, Algebraic K-theory is used copiously to answer these questions. For example, if G is a finite group, T any G-set, $\mathcal{C}$ an exact category, we construct in 10.2 for all $n \geq 0$, equivariant higher K-functors.

$$K_n^G(-, \mathcal{C}, T),\ K_n^G(-, \mathcal{C}, T), K_n^G(-, \mathcal{C})$$

as Mackey functors from the category $GSet$ of G-sets to the category $\mathbb{Z}$-$\mathcal{M}od$ of Abelian groups (i.e. functors satisfying certain functorial properties, in particular, categorical version of Mackey subgroup theorem in representation theory) in such a way that for any subgroup H of G we identify $K_n^G(G/H, \mathcal{M}(R))$ with $K_n(\mathcal{M}(RH)) := G_n(RH), K_n^G(G/H, \mathcal{P}(R))$ with $K_n(\mathcal{P}_R(RH)) := G_n(R, H)$ and $P_n^G(G/H, \mathcal{P}(R), G/e)$ with $K_n(RH)$ for all $n \geq 0$ (see [52, 53]). Analogous constructions are done for profinite group actions (chapter 11) and compact Lie group actions (chapter 12), finite group actions in the context of Waldhausen categories, chapter 13, as well as equivariant homology theories for the actions of discrete groups (see chapter 14).

For such Mackey functors M, one can always find a canonical smallest class U_M of subgroups of G such that the values of M on any G-set can be computed from their restrictions to the full subcategory of G-sets of the form G/H with $H \in U_M$. The computability of the values of M from its restriction to G-sets of the form $G/H, H \in U_M$ for finite and profinite groups is expressed in terms of vanishing theorems for a certain cohomology theory associated with M (U_M) - a cohomology theory which generalizes group cohomology. In 9.2, we discuss the cohomology theory (Amitsur cohomology) of Mackey functors, defined on an arbitrary category with finite coproducts, finite pullbacks and final objects in 9.1 and then specialize as the needs arise for the cases of interest-category of G-sets for G finite (in chapter 9 and chapter 10), G profinite (chapter 11) - yielding vanishing theorems for the cohomology of the K-functors as well as cohomology of profinite groups (11.2) (see [109]).

The equivariant K-theory discussed in this book yields various computations of higher K-theory of grouprings. For example apart from the result that higher K-theory of RG (G finite or compact Lie group) can be computed by restricting to hyper elementary subgroups of G (see 10.4 and 12.3.3) (see [108, 116]), we also show that if R is a field k of characteristic p and G a finite or profinite group, then the Cartan map $K_n(kG) \longrightarrow G_n(kG)$ induces an isomorphism $\mathbb{Z}(\frac{1}{p}) \otimes K_n(kG) \cong \mathbb{Z}(\frac{1}{p}) \otimes G_n(kG)$ leading to the result that for all $n \geq 1$, $K_{2n}(kG)$ is a p-group for finite groups G. We also have an interesting result that if R is the ring of integers in a number field, G a finite group then the Waldhausen K-groups of the category $(Ch_b(\mathcal{M}(RG), \omega)$ of bounded complexes of finitely generated RG-modules with stable quasi-isomorphisms as weak equivalences are finite Abelian groups.

The last chapter (chapter 14) which is devoted to Equivariant homology theories, also aims at computations of higher algebraic K-groups for grouprings of discrete groups via induction techniques also using Mackey functors. In fact, an important criteria for a G-homology theory $\mathcal{H}_n^G : G\mathcal{S}et \longrightarrow \mathbb{Z}\text{-}\mathcal{M}od$ is that it is isomorphic to some Mackey functor: $G\mathcal{S}et \longrightarrow \mathbb{Z}\text{-}\mathcal{M}od$. The chapter is focussed on a unified treatment of Farrell and Baum-Connes isomorphism conjectures through Davis-Lück assembly maps (see 14.2 or [40]) as well as some specific induction results due to W. Lück, A. Bartels and H. Reich (see 14.3 or [14]). One other justification for including Baum-Connes conjecture in this unified treatment is that it is well known by now that Algebraic K-theory and Topological K-theory of stable C^*-algebras do coincide (see [205]). We review the state of knowledge of both conjectures (see 14.3, 14.4) and in the case of Baum-Connes conjecture also discuss its various formulations including the most recent in terms of quantum group actions.

Time, space and the heavy stable homotopy theoretic machinery involved (see [147]) (for which we could not prepare the reader) has prevented us from including a G-spectrum formulation of the equivariant K-theory developed in chapters 10, 11, 12, 13. In [192, 193, 194], K. Shimakawa provided, (for G a finite group) a G-spectrum formulation of part of the (absolute) equivariant theory discussed in 10.1. It will be nice to have a G-spectrum formulation of the relative theory discussed in 10.2 as well as a G-spectrum formulation for the equivariant theory discussed in 11.1 and 12.2 for G profinite and G compact Lie group. However, P. May informs me that equivariant infinite loop space theory itself is only well understood for finite groups. He thinks that profinite groups may be within reach but compact Lie groups are a complete mystery since no progress has been made towards a recognition principle in that case. Hence, there is currently no idea about how to go from the type of equivariant Algebraic K-theory categories defined in this book to a G-spectrum when G is a compact Lie group.

Appendix A contains some known computations while Appendix B consists of some open problems.

The need for this book

1) So far, there is no book on higher Algebraic K-theory of orders and grouprings. The results presented in the book are only available in scattered form in journals and other scientific literature, and there is a need for a coordinated presentation of these ideas in book form.

2) Computations of higher K-theory even of commutative rings (e.g. $\mathbb{Z}$) have been notoriously difficult and up till now the higher K-theory of $\mathbb{Z}$ is yet to be fully understood. Orders and grouprings are usually non-commutative rings that also involve non-commutative arithmetic and computations of higher K-theory of such rings are even more difficult, since methods of etále cohomology etc. do not work. So it is desirable to collect together in book form methods that have been known to work for computations of higher K-theory of such non-commutative rings as orders and grouprings.

3) This is the first book to expose the characterization of all higher algebraic K-theory as Mackey functors leading to equivariant higher algebraic K-theory and their relative generalization, also making computations of higher K-theory of grouprings more accessible. The translation of the abstract topological constructions into representation theoretic language of Mackey functors has simplified the theory some what and it is desirable to have these techniques in book form.

4) Interestingly, obtaining results on higher K-theory of orders Λ (and hence grouprings) for all $n \geq 0$ have made these results available for the first time for some classical K-groups. For instance, it was not known classically that if R is the ring of integers in a number field F, and Λ any R-order in a semi-simple F-algebra, then $K_2(\Lambda), G_2(\Lambda)$ (or even $SK_s(\Lambda), SG_2(\Lambda)$) are finite groups. Having these results for $K_{2n}(\Lambda), G_{2n}(\Lambda)$ and hence $SK_{2n}(\Lambda), SG_{2n}(\Lambda)$ for all $n \geq 1$ makes these results available now for $n = 1$.

5) Also computations of higher K-theory of orders which automatically yield results on higher K-theory of $RG(G$ finite) also extends to results on higher K-theory of some infinite groups e.g. computations of higher K-groups of virtually infinite cyclic groups that are fundamental to the subject.

Who can use this book?
It is expected that readers would already have some working knowledge of algebra in a broad sense including category theory and homological algebra, as well as working knowledge of basic algebraic topology, representation theory, algebraic number theory, some algebraic geometry and operator algebras. Nevertheless, we have tried to make the book as self-contained as possible by defining the most essential ideas.

As such, the book will be useful for graduate students who have completed at least one year of graduate study, professional mathematicians and researchers of diverse backgrounds who want to learn about this subject as well as specialists in other aspects of K-theory who want to learn about this approach to the subject. Topologists will find the book very useful in updating their knowledge of K-theory of orders and grouprings for possible applications and representation theorists will find this innovative approach to and applications of their subject very enlightening and refreshing while number theorists and arithmetic algebraic geometers who want to know more about non-commutative arithmetics will find the book very useful.

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Notes on Notations

Notes on Notation

- $mor_{\mathcal{C}}(A,B), Hom_{\mathcal{C}}(A,B)$:= set of $\mathcal{C}$-morphisms from A to B ($\mathcal{C}$ a category
- $\bar{A} = K(A)$ = Gröthendieck group associated to a semi-group A
- $VB_F(X)$ = category of finite dimensional vector bundles on X ($F = \mathbb{R}$ or $\mathbb{C}$)
- $X(G)$ = set of cyclic quotients of a finite group G
- $\mathcal{A}[z]$ = polynomial extension of an additive category $\mathcal{A}$
- $\mathcal{A}[z, z^{-1}]$, Laurent polynomal extension of $\mathcal{A}$
- $\mathcal{O}r_{\mathcal{F}}(G) = \{G/H | H \in \mathcal{F}\}$, $\mathcal{F}$ a family of subgroups of G
- $A_{R,\mathcal{F}}$ = assembly map
- $\mathcal{H}_G(S,B) := \{G\text{-maps } f : S \to B\}$, S a G-set, B a $\mathbb{Z}G$-module
- $B(S)$ = set of all set-theoretic maps $S \to B$. Note that
- $\mathcal{H}_G(S,B)$ is the subgroup of G-invariant elements in $B(S)$
- $\mathcal{A}(G)$ = category of homogeneous G-spaces (G a compact Lie group)
- $\Omega(\mathcal{B})$:= Burnside ring of a based category $\mathcal{B}$
- $\Omega(G)$:= Burnside ring of a group G
- $\|\mathcal{B}\|$:= Artin index of a based category $\mathcal{B}$
:= exponent of $\bar{\Omega}(\mathcal{B})/\Omega(\mathcal{B})$
- M_m^n := n-dimensional mod-m Moore space
- $H_n(X, \underline{E}) = \underline{E}_n(X)$:= homology of a space X with coefficient in a spectrum $\underline{E}$
- $H^n(X, \underline{E}) = \underline{E}^n(X)$:= cohomology of a space X with coefficients in a spectrum $\underline{E}$
- $\mathcal{P}(A)$:= category of finitely generated projective A-modules (A a ring with identity)