One of a Kind

Calculus: Early Vectors, Preliminary Edition
Volume II

Stewart

Texas A & M

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Calculus: Early Vectors, Preliminary Edition Volume II

Stewart

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7 Techniques of Integration

■ Common integration is only the memory of differentiation. The different devices by which integration is accomplished are changes, not from the known to the unknown, but from forms in which memory will not serve us to those in which it will.

Augustus de Morgan

Because of the Fundamental Theorem of Calculus, we can integrate a function if we know an antiderivative, that is, an indefinite integral. We summarize here the most important integrals that we have learned so far.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1) \qquad \int \frac{1}{x} \, dx = \ln |x| + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C \qquad \int \csc x \cot x \, dx = -\csc x + C$$

$$\int \sinh x \, dx = \cosh x + C \qquad \int \cosh x \, dx = \sinh x + C$$

$$\int \tan x \, dx = \ln |\sec x| + C \qquad \int \cot x \, dx = \ln |\sin x| + C$$

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

In this chapter we develop techniques for using these basic integration formulas to obtain indefinite integrals of more complicated functions. We learned the most important method of integration, the Substitution Rule, in Section 6.5 (in Volume I). The other general technique, integration by parts, is presented in Section 7.1. Then we learn methods that are special to particular classes of functions such as trigonometric functions and rational functions.

Integration is not as straightforward as differentiation; there are no rules that absolutely guarantee obtaining an indefinite integral of a function. Therefore, in Section 7.6 we discuss a strategy for integration.

7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int \left[f'(x)g(x) + f(x)g'(x)\right] dx = f(x)g(x)$$

or

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x)$$

We can rearrange this latter equation as

(1)
$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Formula 1 is called **the formula for integration by parts.** It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

(2)
$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example 1 Find $\int x \sin x\,dx$.

Solution Using Formula 1 Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For g we can choose *any* antiderivative of g'.) Thus, using Formula 1, we have

$$\begin{aligned}\int x\sin x\,dx &= f(x)\,g(x) - \int f'(x)\,g(x)\,dx \\ &= x\,(-\cos x) - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

It is wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

It is helpful to use the pattern:

$$u = \square \qquad dv = \square$$
$$du = \square \qquad v = \square$$

Solution Using Formula 2 Let

$$u = x \qquad dv = \sin x\,dx$$

then

$$du = dx \qquad v = -\cos x$$

$$\text{and so} \quad \int x \sin x\, dx = \int \overbrace{x}^{u}\, \overbrace{\sin x\, dx}^{dv} = \overbrace{x}^{u}\, \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\, \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + C$$

■

Note: Our object in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x \sin x\, dx$ and expressed it in terms of the simpler integral $\int \cos x\, dx$. If we had chosen $u = \sin x$ and $dv = x\, dx$, then $du = \cos x\, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x \sin x\, dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x\, dx$$

But $\int x^2 \cos x\, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for u and dv, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\, dx$ can be readily integrated to give v.

Example 2 Evaluate $\displaystyle\int \ln x\, dx$.

Solution Here we do not have much choice for u and dv. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\, dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x\, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Check the answer by differentiating it.

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than f. ■

Example 3 Find $\displaystyle\int x^2 e^x\, dx$.

Solution Let

$$u = x^2 \qquad dv = e^x\, dx$$

Then

$$du = 2x\, dx \qquad v = e^x$$

Integration by parts gives

(3)
$$\int x^2 e^x\, dx = x^2 e^x - 2\int x e^x\, dx$$

The integral that we obtained, $\int xe^x\,dx$, is simpler than the original integral but is still not obvious. Therefore, we use integration by parts a second time, this time with $u = x$ and $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

Putting this in Equation 3, we get

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2(xe^x - e^x + C) \\ &= x^2e^x - 2xe^x + 2e^x + C_1 \qquad \text{where } C_1 = -2C\end{aligned}$$ ■

Example 4 Evaluate $\int e^x \sin x\,dx$.

Solution Let $u = e^x$ and $dv = \sin x\,dx$. Then $du = e^x\,dx$ and $v = -\cos x$, so integration by parts gives

(4) $$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

Figure 1 illustrates Example 4 by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = \frac{1}{2}e^x(\sin x - \cos x)$. As a visual check on our work, notice that $f(x) = 0$ when F has a maximum or minimum.

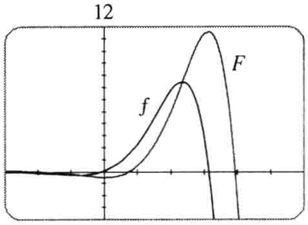


FIGURE 1

The integral that we have obtained, $\int e^x \cos x\,dx$, is no simpler than the original one, but at least it is no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. This time we use $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

(5) $$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x\,dx$, which is where we started. However, if we put Equation 5 into Equation 4 we get

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

This can be regarded as an equation to be solved for the unknown integral. Solving, we obtain

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

and, dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$ ■

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between a and b, assuming f' and g' are continuous, and using the Fundamental Theorem in the form of Equation 6.4.10, we obtain

(6) $$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

Example 5 Calculate $\int_0^1 \tan^{-1} x\,dx$.

Solution Let

$$u = \tan^{-1} x \qquad dv = dx$$

Then $$du = \frac{dx}{1+x^2} \qquad v = x$$

Since $\tan^{-1} x \geq 0$ for $x \geq 0$, the integral in Example 5 can be interpreted as the area of the region shown in Figure 2.

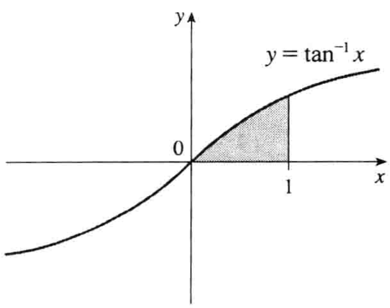


FIGURE 2

So Formula 6 gives

$$\begin{aligned}\int_0^1 \tan^{-1} x\,dx &= x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= 1\cdot\tan^{-1} 1 - 0\cdot\tan^{-1} 0 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx\end{aligned}$$

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = dt/2$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\,dx &= \tfrac{1}{2}\int_1^2 \frac{dt}{t} = \tfrac{1}{2}\ln|t|\Big]_1^2 \\ &= \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2\end{aligned}$$

Therefore $$\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$ ■

Example 6 Prove the reduction formula

(7) $$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

where $n \geq 2$ is an integer.

Solution Let

$$u = \sin^{n-1} x \qquad dv = \sin x\,dx$$

Then $$du = (n-1)\sin^{n-2} x\cos x\,dx \qquad v = -\cos x$$

so integration by parts gives

$$\int \sin^n x\,dx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \qquad \blacksquare$$

The reduction Formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

Exercises 7.1

1–30 ■ Evaluate the integral.

1. $\int xe^{2x}\,dx$

2. $\int x\cos x\,dx$

3. $\int x\sin 4x\,dx$

4. $\int x\ln x\,dx$

5. $\int x^2\cos 3x\,dx$

6. $\int x^2\sin 2x\,dx$

7. $\int (\ln x)^2\,dx$

8. $\int \sin^{-1} x\,dx$

9. $\int \theta\sin\theta\cos\theta\,d\theta$

10. $\int \theta\sec^2\theta\,d\theta$

11. $\int t^2\ln t\,dt$

12. $\int t^3e^t\,dt$

13. $\int e^{2\theta}\sin 3\theta\,d\theta$

14. $\int e^{-\theta}\cos 3\theta\,d\theta$

15. $\int y\sinh y\,dy$

16. $\int y\cosh ay\,dy$

17. $\int_0^1 te^{-t}\,dt$

18. $\int_1^4 \sqrt{t}\ln t\,dt$

19. $\int_0^{\pi/2} x\cos 2x\,dx$

20. $\int_0^1 x^2e^{-x}\,dx$

21. $\int_0^{1/2} \cos^{-1} x\,dx$

22. $\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx$

23. $\int \cos x\ln(\sin x)\,dx$

24. $\int x^3e^{x^2}\,dx$

25. $\int (2x+3)e^x\,dx$

26. $\int x5^x\,dx$

27. $\int \cos(\ln x)\,dx$

28. $\int x\tan^{-1} x\,dx$

29. $\int_1^4 \ln \sqrt{x}\, dx$

30. $\int \sin(\ln x)\, dx$

31–34 ■ First make a substitution and then use integration by parts to evaluate the integral.

31. $\int \sin \sqrt{x}\, dx$

32. $\int x^5 \cos(x^3)\, dx$

33. $\int x^5 e^{x^2}\, dx$

34. $\int_1^4 e^{\sqrt{x}}\, dx$

35–36 ■ Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

35. $\int x \cos \pi x\, dx$

36. $\int \sqrt{x} \ln x\, dx$

37. **(a)** Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\, dx$.

38. **(a)** Prove the reduction formula

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\, dx$.

(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\, dx$.

39. **(a)** Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

where $n \geq 2$ is an integer.

(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\, dx$ and $\int_0^{\pi/2} \sin^5 x\, dx$.

(c) Use part (a) to show that, for odd powers of sin,

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

40. Prove that, for even powers of sin,

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}\,\frac{\pi}{2}$$

41–44 ■ Use integration by parts to prove the reduction formula.

41. $\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$

42. $\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx$

43. $\int (x^2 + a^2)^n\, dx$

$$= \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (x^2+a^2)^{n-1}\, dx$$

$(n \neq -\frac{1}{2})$

44. $\int \sec^n x\, dx = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\, dx$

$(n \neq 1)$

45. Use Exercise 41 to find $\int (\ln x)^3\, dx$.

46. Use Exercise 42 to find $\int x^4 e^x\, dx$.

47–48 ■ Find the area of the region bounded by the given curves.

47. $y = \sin^{-1} x, \quad y = 0, \quad x = 0.5$

48. $y = 5 \ln x, \quad y = x \ln x$

49–50 ■ Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

49. $y = x^2, \quad y = xe^{-x/2}$

50. $y = x^2 - 5, \quad y = \ln x$

51. A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

52. If $f(0) = g(0) = 0$, show that

$$\int_0^a f(x)g''(x)\, dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\, dx$$

53. Use integration by parts to show that

$$\int f(x)\, dx = xf(x) - \int xf'(x)\, dx$$

54. If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\, dy$$

[*Hint:* Use Exercise 53 and make the substitution $y = f(x)$.]

55. Use Exercise 54 to evaluate $\int_1^e \ln x\, dx$.

56. In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of Exercise 54.

57. Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$.

(a) Show that $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$.

(b) Use Exercise 40 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.

(d) Use part (c) and Exercises 39 and 40 to show that

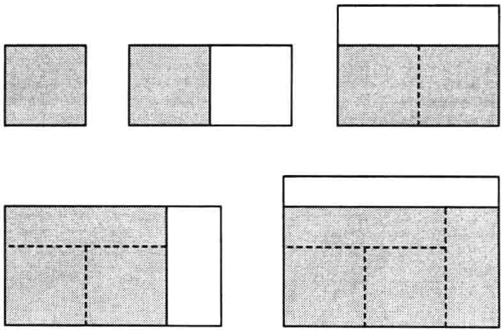
$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

and is called the *Wallis product.*

(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle. (See the figure.) Find the limit of the ratios of width to height of these rectangles.



7.2 Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

Example 1 Evaluate $\int \cos^3 x\,dx$.

Solution Here the appropriate identity is $\cos^2 x = 1 - \sin^2 x$. We write

$$\cos^3 x = \cos^2 x \cdot \cos x = (1 - \sin^2 x)\cos x$$

It is useful to have the extra factor of $\cos x$ because if we make the substitution $u = \sin x$, then we have $du = \cos x\,dx$. Thus

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x \cdot \cos x\,dx = \int (1 - \sin^2 x)\cos x\,dx \\ &= \int (1 - u^2)\,du = u - \tfrac{1}{3}u^3 + C \\ &= \sin x - \tfrac{1}{3}\sin^3 x + C\end{aligned}$$

■

Figure 1 shows the graphs of the integrand $\sin^5 x \cos^2 x$ in Example 2 and its indefinite integral (with $C = 0$). Which is which?

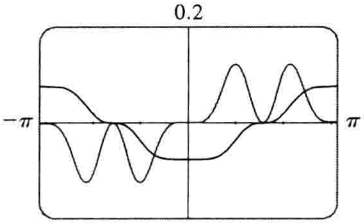


FIGURE 1

The method used in Example 1 suggests the following general strategy to be used in evaluating integrals of the form $\int \sin^m x \cos^n x\,dx$, where $m \ge 0$ and $n \ge 0$ are integers and either m or n is odd.

HOW TO EVALUATE $\int \sin^m x \cos^n x\,dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x\,dx &= \int \sin^m x(\cos^2 x)^k \cos x\,dx \\ &= \int \sin^m x(1 - \sin^2 x)^k \cos x\,dx\end{aligned}$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x\,dx &= \int (\sin^2 x)^k \cos^n x \sin x\,dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x\,dx\end{aligned}$$

Then substitute $u = \cos x$.

Example 2 Find $\int \sin^5 x \cos^2 x\,dx$.

Solution Here the power of sine is odd and so we proceed as in case (b), substituting $u = \cos x$:

$$\begin{aligned}\int \sin^5 x \cos^2 x\, dx &= \int \sin^4 x \cos^2 x \sin x\, dx \\ &= \int (1-\cos^2 x)^2 \cos^2 x \sin x\, dx \\ &= \int (1-u^2)^2 u^2 (-du) \\ &= -\int (u^2 - 2u^4 + u^6)\, du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C\end{aligned}$$ ■

The advice in (a) and (b) works if either sine or cosine has an odd exponent. In the remaining case (both m and n are even), we proceed as follows.

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

Example 3 shows that the area of the region shown in Figure 2 is $\pi/2$.

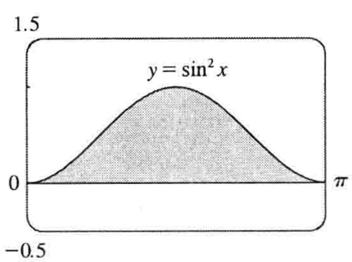


FIGURE 2

Example 3 Evaluate $\int_0^\pi \sin^2 x\, dx$.

Solution Here $m = 2$ and $n = 0$, so we use the half-angle formula for $\sin^2 x$:

$$\begin{aligned}\int_0^\pi \sin^2 x\, dx &= \tfrac{1}{2}\int_0^\pi (1-\cos 2x)\, dx = \left[\tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x)\right]_0^\pi \\ &= \tfrac{1}{2}(\pi - \tfrac{1}{2}\sin 2\pi) - \tfrac{1}{2}(0 - \tfrac{1}{2}\sin 0) = \tfrac{1}{2}\pi\end{aligned}$$

Notice that we mentally made the substitution $u = 2x$ when integrating $\cos 2x$. Another method for evaluating this integral was given in Exercise 37 in Section 7.1. ■

Example 4 Find $\int \sin^4 x\, dx$.

Solution It is possible to evaluate this integral using the reduction formula for $\int \sin^n x\, dx$ (Equation 7.1.7) together with Example 1 (as in Exercise 37 in Section 7.1), but another method is to write $\sin^4 x = (\sin^2 x)^2$ and use (c):

$$\begin{aligned}\int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\ &= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \\ &= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx\end{aligned}$$

Since $\cos^2 2x$ occurs, we must use another half-angle formula

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x)$$

This gives

$$\begin{aligned}\int \sin^4 x\,dx &= \tfrac{1}{4}\int [1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]\,dx \\ &= \tfrac{1}{4}\int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx \\ &= \tfrac{1}{4}\left(\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right) + C\end{aligned}$$ ■

Integrals of the form $\int \tan^m x \sec^n x\,dx$ can be integrated in the following cases.

HOW TO EVALUATE $\int \tan^m x \sec^n x\,dx$

(a) If the power of secant is even ($n = 2k$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x\,dx &= \int \tan^m x(\sec^2 x)^{k-1}\sec^2 x\,dx \\ &= \int \tan^m x(1 + \tan^2 x)^{k-1}\sec^2 x\,dx\end{aligned}$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x\,dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\,dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\,dx\end{aligned}$$

Then substitute $u = \sec x$.

Example 5 Evaluate $\displaystyle\int \tan^6 x \sec^4 x\,dx$.

Solution Since the secant has an even exponent, we factor $\sec^2 x$ from the integrand and substitute $u = \tan x$ so that $du = \sec^2 x\,dx$. The rest of the integrand is then expressed completely in terms of $\tan x$ by means of the identity $\sec^2 x = 1 + \tan^2 x$:

$$\begin{aligned}\int \tan^6 x \sec^4 x\,dx &= \int \tan^6 x \sec^2 x \sec^2 x\,dx \\ &= \int \tan^6 x(1 + \tan^2 x)\sec^2 x\,dx \\ &= \int u^6(1 + u^2)\,du = \int (u^6 + u^8)\,du \\ &= \frac{u^7}{7} + \frac{u^9}{9} + C \\ &= \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x + C\end{aligned}$$ ■