Foundations of Mathematics

WITH APPLICATION TO THE SOCIAL AND MANAGEMENT SCIENCES

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FOUNDATIONS OF MATHEMATICS

To Rachael and Ted

PREFACE

This textbook was developed in response to a need in a particular situation, as is the case with most textbooks. Many students came to Kent State University with two or more years of high school mathematics to their credit, but they had forgotten much of it or had not considered it of prime importance when they had studied it. Consequently, they were poorly prepared for college mathematics courses and for mathematically oriented courses in the management and social sciences. It is for such students that we wrote this text.

The selection of topics was made to include both those that are foundational and those that are of interest to business and social science students. The point of view of the authors is that, while the mathematical concepts are most important, the usefulness of these concepts should also be stressed. We have therefore used many examples and interspersed a large number of applications in the mathematical development. The level at which a particular topic begins is somewhat elementary but progresses to concepts of proper sophistication at the college level.

We have tried to make the material readable for the student and at the same time preserve mathematical preciseness. This necessarily has led to some compromises, but we believe that "Foundations of Mathematics" will be both instructional and interesting to students with two or more years of high-school mathematics who are pursuing nonscience courses of study.

The text contains more than enough material for 6 semester hours or 10 quarter hours of course work. Parts of the material are not in the mainstream of the development and may be omitted. Chapters 1, 2, 3, 8, 10, 12, 13, and 14 can be considered a basic course in applications of the mathematics of the real-number system; this is sufficient material for 5 semester hours. Additional hours should include Chaps. 4 and 5. One or more of Chaps. 6, 7, 9, and 11 may be introduced, depending upon the applications and depth of topics desired.

The credit for the development of this text should include the many students whose inquiring minds aided the authors immensely. We also thank the College of Business Administration at Kent State University for their foresight and cooperation and Mr. Ronald Ehresman, Mr. Byron Dressler, Mrs. Sandra Basinger, and Miss Anita Brower for their contributions to the success of this project. We are especially grateful to Mrs. H. N. Abhau for her patience in typing the manuscript.

Grace A. Bush
John E. Young

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1 Sets and Subsets

1.1 INTRODUCTION

It is assumed that a reader of this book has had several years of mathematical training in high school. It is quite possible that in all this experience the reader was not introduced to the notion of a set, or if he was given an introduction to sets, it was as a subject apart from the mainstream of mathematics. In this text, set theory and the accompanying symbolism will be used as a base for developing most of the material presented. It is particularly important, then, that this first chapter be read carefully and understood before successive chapters are studied. Success or failure in understanding many of the concepts presented in the text will depend upon the ability of the student to apply the definitions and theorems presented here.

1.2 SETS

A set is thought of as a collection or group of distinct elements. A set can be specified in one of two ways—by listing the members of the collection or by specifying what it is that the members of the collection have in common. Some examples of sets are:

1. the set of directors of a corporation
2. the set of accounts receivable of a company
3. the set of even numbers
4. the set of paintings in the Louvre
5. the set of people who have incomes in excess of $100,000,000 a year
6. the set of numbers which can be substituted for x so that $x + 7 = 10$
7. the set A, B, C
8. the set of airplanes built by Boeing Aircraft before 1900

Only example 7 lists the elements of the set. The rest of the examples describe a set by supplying a rule, or measure, for determining whether or not a given object belongs to the set. It is important to understand that:

1. *A set must be well defined.* It must be possible to tell whether a given element belongs to a set either by checking it against the list of elements of the set or by deciding whether it satisfies or does not satisfy the rule governing membership for that set. Rules governing set membership must make perfectly clear what elements are included in a particular set. The set of "all nice people" is not well defined as there is no criterion given for deciding whether a person is "nice."
2. *The elements in a set are distinct.* If an object is listed as an element of a set it should not be listed a second time. The set of letters necessary to write the word "walk" contains w, a, l, and k. The set of letters needed to write "teeth" contains only t, e, and h.
3. *The order of the elements in a list is not significant.* The set containing the elements 1, 2, 3 is exactly the same as the set containing 2, 3, 1.

1.3 SET NOTATION

Several symbols are used for denoting sets. One of these is the brace { }. The set whose members or elements are 1, 2, 3, 4 would be indicated by

$$\{1,2,3,4\}$$

This same set can be denoted by writing

$$\{n \mid n \text{ is a counting number less than five}\}$$

This is read "the set of all n such that n is a counting number less than five." In the first case the elements of the set were listed, and in the second case they were described.

set notation

Capital letters are used to indicate or name sets. For example, C might represent the set of chairs in a room, P the set of people majoring in economics. Small letters are used to represent the elements of a set. For example, r might represent a rocking chair in C, or p_1 a particular person in P. To say that an object or element is a member of a set means that the element has some characterizing property which places it in the set. To denote that the element r belongs to the set C, we write $r \in C$. This is read "r is an element of C." If there is a slash through the symbol, $\notin$, it is read "is not an element of." This is consistent with the mathematical convention of negating a symbol with a slash mark. Thus $\neq$ is read "is not equal to." Examine the following list of statements. Make sure you understand why each statement was made.

1. $2 \in \{1,2,3,4\}$
2. $\{4\} \notin \{1,2,3,4\}$
3. AT&T $\in \{y \mid y$ is a stock traded on the New York Stock Exchange$\}$
4. George Washington $\in \{x \mid x$ served as President of the United States$\}$

In statement 2 the symbol $\notin$ is used. It is true that $4 \in \{1,2,3,4\}$ but statement 2 does not say this.

1.4 SUBSETS

Consider the set $S = \{1,3,5,8,10\}$. The set $A = \{5,8,10\}$ consists solely of elements selected from the set S. It is possible to write many other sets which consist only of elements from S. For example, $B = \{3\}$ and $C = \{3,5,8,10\}$ satisfy this condition. A, B, and C are called *subsets* of S.

subset

Definition The set A is a *subset* of the set B if every element in A is an element of B. Symbolically this is expressed $A \subseteq B$ and read "A is a subset of B." This same relation can be expressed as $B \supseteq A$, which says "B contains A as a subset."

According to the definition, the set $K = \{1,3,5,8,10\}$ is a subset of $S = \{1,3,5,8,10\}$. K happens to contain *all* the elements of S, but there is nothing in the definition that says that this is not allowed. To differentiate between subsets of a given set which contain fewer elements than the given set and subsets which contain as many, a second definition is used.

proper subset

Definition The set A is a *proper subset* of the set B if every element of A is an element of B but not every element of B is an element of A. The notation $A \subset B$ denotes that A is a proper subset of B.

If set $A = \{x \mid x$ is a truck belonging to the Big Freight Company$\}$ and set $B = \{y \mid y$ is an asset of the Big Freight Company$\}$, the notation $A \subset B$ expresses the idea that the trucks of the company are *part* (but not all) of the assets.

universal set

Definition The set containing the totality of elements for any particular discussion or situation is called the *universal set* and is denoted by the symbol U.

In the accounting department of any company there will be a set of Accounts Receivable, a set of Accounts Payable, a set of Fixed Assets, etc. All these different sets would be subsets of the universal set, U, of all the accounts of the company.

Frequently sets are defined by stating a property that every element in the set must possess. It is also possible to state a property which none of the objects considered have. For example, consider the set of people who became President of the United States on their thirty-sixth birthdays. Investigation will show that no person fulfills the criterion for membership in the set. In this case the set contains no elements.

null set

Definition A set which contains no elements is an *empty* or *null* set.

The null set could be indicated by a pair of empty braces, { }, but experience has shown that this is a troublesome notation. Therefore, the empty set is denoted by the symbol $\emptyset$. This symbol is written without braces. There is only one empty set and the same empty set is a subset of every set.

Exercise 1A

1. If $A = \{1,2,3\}$, is A a subset of A according to the definition?
2. For any set S is $S \subseteq S$? Is $S \subset S$?
3. Which of the following are empty sets?
   (*a*) The set of women who are heads of governments
   (*b*) The set of numbers used in counting
   (*c*) The set of companies with assets of over \$100,000,000,000,000,000
   (*d*) The set of stocks on the New York Stock Exchange that closed at $7\frac{1}{2}$ yesterday
4. (*a*) Is $0 \in \emptyset$?
   (*b*) If S is a set, is $\emptyset$ a subset of S?
   (*c*) Is $8 \in \{8,9,10\}$?
   (*d*) Is $\{2\} \in \{1,2,3\}$?
5. If $P = \{\text{Jones, Brown, Deere, Hall}\}$, then:
   (*a*) Is Jones $\in P$?
   (*b*) Is Brewster $\in P$?
   (*c*) Is $\{\text{Jones, Hall}\} \subset P$?
   (*d*) Is $\{\text{Jones, Hall, Deere, Brown}\} \subset P$?
   (*e*) Is $\emptyset \subset P$?
6. Write the elements of

(*a*) the set of whole numbers from 8 through 15
(*b*) the cars manufactured at present by Ford Division of Ford Motor Company
(*c*) the vowels in the alphabet which are between j and n (consider the alphabet in order)
(*d*) the subjects you are studying this term

7. Let $A = \{x \mid x$ is an insurance salesman$\}$ and $B = \{y \mid y$ is an insurance salesman who sold over \$1,000,000 worth of insurance last year$\}$. What does it mean to say $B \subset A$? How does this differ from saying $B \subseteq A$?
8. (*a*) List all the subsets of $A = \{a,b\}$.
   (*b*) List all the subsets of {doctor, lawyer, baker}.
   (*c*) List all the subsets of $\{a,b,c,d\}$. There should be 16.
   (*d*) If a set has n elements, what formula expresses the total number of subsets of the given set?
9. Write a universal set which would contain all the following as subsets:
   (*a*) $\{a,e,i,o,u\}$, $\{a,b,c\}$, $\{x,y,z\}$
   (*b*) $\{1,2,3,4\}$, $\{2,4,6,8\}$, $\{1,3,5,9\}$
   (*c*) {red, blue, yellow}, {blue, pink, white}, {green}
10. If set A has 31 proper subsets, how many elements are in set A?

1.5 EQUAL AND EQUIVALENT SETS

A young man settles himself to the task of paying the bills which have accumulated over the past week. The set of bills to be paid is $A =$ {ABC Light and Power, First National Bank (house payment), XYZ Finance (car payment), SOS Oil Company}. Since each of the bills is from a different company, if the man wishes to settle all of them he will have to write checks to each of the different concerns. The set of companies from which he has received bills is exactly the same as the set of companies to which he plans to send payment checks. These two sets are said to be *identical* or *equal.*

identical sets

Definition Two sets are *identical,* or *equal,* if and only if every element of each set is also an element of the other set. If two sets are equal we write $A = B$. For A to equal B, we must have $A \subseteq B$ and $B \subseteq A$.

Assume that the checks the man wrote were each for a different amount. The set of checks is not equal to the set of companies from which he has received bills; however, it is possible to match each bill with just one check and each check with just one bill. Every time two sets are such that every element of the first set can be paired with just one element of the second set and vice versa we say a *one-to-one correspondence* has been established between the two sets. In this example a one-to-one correspondence was established between the checks and the creditors. The set of checks and the set of creditors are *equivalent sets.*

equivalent sets

Definition Two sets which can be put into one-to-one correspondence with one another are said to be *equivalent*.

The sets $\{a,b,c,d\}$ and $\{\#,\&,\$,\%\}$ are equivalent since a one-to-one correspondence can be established between them. It is not necessary to know how many elements are in each set to prove that they are equivalent; it is only necessary to match the elements of one set, one by one, with the elements of the other.

Primitive man kept track of his flocks of sheep by putting into a container a small stone for each sheep he let out to graze in the morning and removing from the container a stone for each sheep let back into the pen at night. If a stone or two remained in the container after the evening tally, he would know he must go and search for lost lambs. Modern department stores sometimes use the same technique in women's departments. A woman desiring to try on three dresses in dressing room 4 stops at a desk where a clerk puts three rings on a hook under No. 4 on a tally board. After trying on the dresses, the woman again stops at the desk and the clerk removes from the hook under No. 4 the number of rings which corresponds to the number of garments the woman now returns. If any rings are left on the hook, it is time to call the house detective! In both cases, a one-to-one correspondence of two sets is used to establish equivalence.

1.6 FINITE AND NONFINITE SETS

If a student wants to know how many pairs of slacks he owns, he "counts" them; that is, he sets up a one-to-one correspondence between the set of slacks and the set of counting numbers beginning with 1 and named in order. If he counts to five in his matching, then he has corresponded his slacks with the set

$$\{1,2,3,4,5\}$$

and he knows how many elements there are in his set of slacks.

In any one-to-one correspondence if a given set is matched with the counting numbers 1, 2, 3, . . .,k (where the notation . . . indicates continuing in the same manner up through the term labeled k), then the given set contains k elements. For example, if the set of fingers on a person's hands is matched with the counting numbers, a finger is matched with each number counted. Counting aloud one would say, "One, two, three, four, five, six, seven, eight, nine, ten." The set of fingers contains 10 elements.

finite sets

Definition If a set can be placed in one-to-one correspondence with the first k counting numbers, the set is said to be *finite*.

Example 1.1 The set of letters of the alphabet can be corresponded with the set $\{1,2,3, \ldots,25,26\}$. Since the set of letters in the alphabet was corresponded to the set of the first 26 counting numbers, the set of letters in the alphabet is finite and contains 26 elements.

Example 1.2 The set of citizens of the United States can be corresponded with the set $\{1,2,3, \ldots,k\}$. The value of k is determined by a census every 10 years. This set is also a finite set as it can be corresponded with the set of the first k counting numbers.

infinite sets

Some sets are not finite. Not-finite sets, called *infinite* sets, have some properties quite different from those of finite sets

Example 1.3 The set of all positive fractions with numerators of 1, $A = \{1,\frac{1}{2},\frac{1}{3},\frac{1}{4}, \ldots,1/n, \ldots\}$, cannot be corresponded with a set $N = \{1,2,3, \ldots,k\}$ no matter how large a number k is. If $k = 1{,}000{,}000$, there will be elements in the set A which would not be corresponded with elements from N. If $k = 1{,}000{,}000 \times 1{,}000{,}000$ the same situation will still exist.

Example 1.4 The set of counting numbers $A = \{1,2,3, \ldots,n, \ldots\}$ has as a subset the set

$$B = \{10,20,30, \ldots,10n, \ldots\}$$

Each element from A can be matched to a single element of B by corresponding it to the element 10 times as large in B, and each element of B can be matched to the element one-tenth as large in A. A one-to-one correspondence has been established between the set A and a proper subset of the set A.

This correspondence of a set with a proper subset of itself can be made only when the sets are infinite. A somewhat startling conclusion is that the sets A and B have the same number of elements.

Exercise 1B

1. Decide in each case whether set A is identical to set B:
   (*a*) $A = \{1,2,3\}$ $B = \{1,2,3,4\}$
   (*b*) $A = \{\text{Jones, Wilson, Brown}\}$ $B = \{\text{Brown, Jones, Wilson}\}$
   (*c*) $A = \{2\}$ $B = \{x \mid 2x = 4\}$
   (*d*) $A = \{c,d\}$ $B = \{C,D\}$
   (*e*) $A = \{0\}$ $B = \emptyset$
   (*f*) $A = \{0\}$ $B = \{\emptyset\}$
   (*g*) $A = \{\#,\$,\%,\&\}$ $B = \{\%,\&,\#,\$\}$
2. In problem 1, which of the sets are equivalent?

3. Let $A = \{\$,@,\%,¢\}$. Write two sets which are equal to A.
4. Can a one-to-one correspondence be established between your fingers and your toes?
5. Are two identical sets equivalent? Are two equivalent sets identical?
6. Can a one-to-one correspondence be established between items kept in stock in a store and the prices of these items? Between the prices and the price tags? When?
7. Indicate for each of the following whether the set is finite, infinite, or empty:
   (*a*) The set of cars manufactured by Ford Motor Company
   (*b*) The set of all snowflakes that have fallen since the creation of the earth
   (*c*) The set of United States astronauts who have gone to Mars
   (*d*) The set of women graduates of the Air Force Academy
   (*e*) The set of numbers divisible by 5
8. Show that the set $\{a,b,c,d,e\}$ is not infinite.
9. Is the set of letters necessary to spell the word "chess" equal to the set of letters needed to spell the word "cheese?"
10. If you are told that $A \subseteq B$, does that also tell you that A and B are equivalent?
11. If you are told that $A \subseteq B$ and that $B \subseteq A$, does this also tell you that A and B are equivalent? Does it tell you any more about A and B?

1.7 BASIC OPERATIONS ON SETS

Given the sets A and B of a universal set U (that is, the set U contains A and B as subsets), new sets can be constructed by the application of certain operations to the sets A, B, and U. Before going further it is necessary to understand what is meant by an operation. You are already acquainted with such operations as addition, subtraction, and multiplication on the set of numbers. Operations in arithmetic are rules by which one or more numbers are associated with a single number. For example, if the numbers are chosen from the set of positive whole numbers, the operation "addition" associates the number 7 with the pair of numbers 3 and 4. We usually write $3 + 4 = 7$ where the $+$ is a symbol used to represent the operation of addition. Similarly, operations on sets are rules which associate one set or several sets with another set.

If the set of directors of the Fast Corporation is $A = \{$Green, White, Brown, Black$\}$ and the set of directors of the Slow Corporation is $B = \{$Black, Brown, Short, Long$\}$, a set which contains those people who are directors of either the Fast Corporation or the Slow Corporation or both is $C = \{$Green, White, Brown, Black, Short, Long$\}$. A new set has been formed; each element in this set is in A or in B or in both. This new set which is associated with the pair of sets A and B is called the *union* of A and B. The operation of unioning is represented by the symbol $\cup$. That is, $A \cup B = \{$Green, White, Brown, Black, Short, Long$\}$.

Union *Definition* The *union* of two sets A and B is the set which contains all the elements which are in A or in B or in both.