DISCRETE MATHEMATICS AND ITS APPLICATIONS
Series Editor KENNETH H. ROSEN

CHROMATIC GRAPH THEORY

GARY CHARTRAND
PING ZHANG

CRC Press
Taylor & Francis Group
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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-13: 978-1-58488-800-0 (Hardcover)

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PREFACE

Beginning with the origin of the Four Color Problem in 1852, the field of graph colorings has developed into one of the most popular areas of graph theory. This book introduces graph theory with a coloring theme. It explores connections between major topics in graph theory and graph colorings, including Ramsey numbers and domination, as well as such emerging topics as list colorings, rainbow colorings, distance colorings related to the Channel Assignment Problem, and vertex/edge distinguishing colorings. Discussions of a historical, applied, and algorithmic nature are included. Each chapter in the text contains many exercises of varying levels of difficulty. There is also an appendix containing suggestions for study projects.

The authors are honored to have been invited by CRC Press to write a textbook on graph colorings. With the enormous literature that exists on graph colorings and the dynamic nature of the subject, we were immediately faced with the challenge of determining which topics to include and, perhaps even more importantly, which topics to exclude. There are several instances when the same concept has been studied by different authors using different terminology and different notation. We were therefore required to make a number of decisions regarding terminology and notation. While nearly all mathematicians working on graph colorings use positive integers as colors, there are also some who use nonnegative integers. There are instances when colorings and labelings have been used synonymously. For the most part, colorings have been used when the primary goal has been either to minimize the number of colors or the largest color (equivalently, the span of colors).

We decided that this book should be intended for one or more of the following purposes:

- a course in graph theory with an emphasis on graph colorings, where this course could be either a beginning course in graph theory or a follow-up course to an elementary graph theory course,
- a reading course on graph colorings,
- a seminar on graph colorings,
- as a reference book for individuals interested in graph colorings.

To accomplish this, it has been our goal to write this book in an engaging, student-friendly style so that it contains carefully explained proofs and examples and contains many exercises of varying difficulty.

This book consists of 15 chapters (Chapters 0-14). Chapter 0 provides some background on the origin of graph colorings – primarily giving a discussion of the Four Color Problem. For those readers who desire a more extensive discussion of the history and solution of the Four Color Problem, we recommend the interesting book by Robin Wilson, titled *Four Colors Suffice: How the Map Problem Was Solved*, published by Princeton University Press in 2002.

To achieve the goal of having the book self-contained, Chapters 1-5 have been written to contain many of the fundamentals of graph theory that lie outside of graph colorings. This includes basic terminology and results, trees and connectivity, Eulerian and Hamiltonian graphs, matchings and factorizations, and graph embeddings. The remainder of the book (Chapters 6-14) deal exclusively with graph colorings. Chapters 6 and 7 provide an introduction to vertex colorings and bounds for the chromatic number. The emphasis of Chapter 8 is vertex colorings of graphs embedded on surfaces. Chapter 9 discusses a variety of restricted vertex colorings, including list colorings. Chapter 10 introduces edge colorings, while Chapter 11 discusses monochromatic and rainbow edge colorings, including an introduction to Ramsey numbers. Chapter 11 also provides a discussion of the Road Coloring Problem. The main emphasis of Chapter 12 is complete vertex colorings. In Chapter 13, several distinguishing vertex and edge colorings are described. In Chapter 14 many distance-related vertex colorings are introduced, some inspired by the Channel Assignment Problem, as well as a discussion of domination in terms of vertex colorings.

There is an Appendix listing fourteen topics for those who may be interested in pursuing some independent study. There are two sections containing references at the end of the book. The first of these, titled *General References*, contains a list of references, both for Chapter 0 and of a general nature for all succeeding chapters. The second such section (*Bibliography*) primarily contains a list of publications to which specific reference is made in the text. Finally, there is an Index of Names, listing individuals referred to in this book, an Index of Mathematical Terms, and a List of Symbols.

There are many people we wish to thank. First, our thanks to mathematicians Ken Appel, Tiziana Calamoneri, Nicolaas de Bruijn, Ermelinda DeLaViña, Stephen Locke, Staszek Radziszowski, Edward Schmeichel, Robin Thomas, Olivier Togni, and Avraham Trahtman for kindly providing us with information and communicating with us on some topics. Thank you as well to our friends Shashi Kapoor and Al Polimeni for their interest and encouragement in this project. We especially want to thank Bob Stern, Executive Editor of CRC Press, Taylor & Francis Group, for his constant communication, encouragement, and interest and for suggesting this writing project to us. Finally, we thank Marsha Pronin, Project Coordinator, Samantha White, Editorial Assistant, and Jim McGovern, Project Editor for their cooperation.

G.C. & P.Z.

To Frank Harary (1921–2005)

for inspiring so many, including us,

to look for the beauty in graph theory.

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Chapter 0

The Origin of Graph Colorings

If the countries in a map of South America (see Figure 1) were to be colored in such a way that every two countries with a common boundary are colored differently, then this map could be colored using only four colors. Is this true of every map?

While it is not difficult to color a map of South America with four colors, it is not possible to color this map with less than four colors. In fact, every two of Brazil, Argentina, Bolivia, and Paraguay are neighboring countries and so four colors are required to color only these four countries.

It is probably clear why we might want two countries colored differently if they have a common boundary – so they can be easily distinguished as different countries in the map. It may not be clear, however, why we would think that four colors would be enough to color the countries of every map. After all, we can probably envision a complicated map having a large number of countries with some countries having several neighboring countries, so constructed that a great many colors might possibly be needed to color the entire map. Here we understand neighboring countries to mean two countries with a boundary *line* in common, not simply a single point in common.

While this problem may seem nothing more than a curiosity, it is precisely this problem that would prove to intrigue so many for so long and whose attempted solutions would contribute so significantly to the development of the area of mathematics known as Graph Theory and especially to the subject of graph colorings: Chromatic Graph Theory. This map coloring problem would eventually acquire a name that would become known throughout the mathematical world.

The Four Color Problem *Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary are colored differently?*

Figure 1: Map of South America

Many of the concepts, theorems, and problems of Graph Theory lie in the shadows of the Four Color Problem. Indeed ...

Graph Theory is an area of mathematics whose past is always present.

Since the maps we consider can be real or imagined, we can think of maps being divided into more general regions, rather than countries, states, provinces, or some other geographic entities.

So just how did the Four Color Problem begin? It turns out that this question has a rather well-documented answer. On 23 October 1852, a student, namely Frederick Guthrie (1833–1886), at University College London visited his mathematics professor, the famous Augustus De Morgan (1806–1871), to describe an apparent mathematical discovery of his older brother Francis. While coloring the counties of a map of England, Francis Guthrie (1831–1899) observed that he could color them with four colors, which led him to conjecture that no more than four colors would be needed to color the regions of any map.

The Four Color Conjecture *The regions of every map can be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently.*

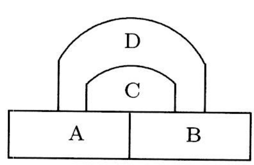
Two years earlier, in 1850, Francis had earned a Bachelor of Arts degree from University College London and then a Bachelor of Laws degree in 1852. He would later become a mathematics professor himself at the University of Cape Town in South Africa. Francis developed a lifelong interest in botany and his extensive collection of flora from the Cape Peninsula would later be placed in the Guthrie Herbarium in the University of Cape Town Botany Department. Several rare species of flora are named for him.

Francis Guthrie attempted to prove the Four Color Conjecture and although he thought he may have been successful, he was not completely satisfied with his proof. Francis discussed his discovery with Frederick. With Francis' approval, Frederick mentioned the statement of this apparent theorem to Professor De Morgan, who expressed pleasure with it and believed it to be a new result. Evidently Frederick asked Professor De Morgan if he was aware of an argument that would establish the truth of the theorem.

This led De Morgan to write a letter to his friend, the famous Irish mathematician Sir William Rowan Hamilton (1805–1865) in Dublin. These two mathematical giants had corresponded for years, although apparently had met only once. De Morgan wrote (in part):

> *My dear Hamilton:*
>
> *A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary lines are differently coloured – four colours may be wanted but not more – the following is his case in which four are wanted.*

A B C D are
names of
colours



> *Query cannot a necessity for five or more be invented . . .*
>
> *My pupil says he guessed it colouring a map of England . . . . The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did . . .*

In De Morgan's letter to Hamilton, he refers to the "Sphynx" (or Sphinx). While the Sphinx is a male statue of a lion with the head of a human in ancient Egypt which guards the entrance to a temple, the Greek Sphinx is a female creature of bad luck who sat atop a rock posing the following riddle to all those who pass by:

> *What animal is that which in the morning goes on four feet, at noon on two, and in the evening upon three?*

Those who did not solve the riddle were killed. Only Oedipus (the title character in *Oedipus Rex* by Sophocles, a play about how people do not control their own destiny) answered the riddle correctly as "Man", who in childhood (the morning of life) creeps on hands and knees, in manhood (the noon of life) walks upright, and in old age (the evening of life) walks with the aid of a cane. Upon learning that her riddle had been solved, the Sphinx cast herself from the rock and perished, a fate De Morgan had envisioned for himself if his riddle (the Four Color Problem) had an easy and immediate solution.

In De Morgan's letter to Hamilton, De Morgan attempted to explain why the problem appeared to be difficult. He followed this explanation by writing:

> *But it is tricky work and I am not sure of all convolutions – What do you say? And has it, if true been noticed?*

Among Hamilton's numerous mathematical accomplishments was his remarkable work with quaternions. Hamilton's quaternions are a 4-dimensional system of numbers of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$. When $c = d = 0$, these numbers are the 2-dimensional system of complex numbers; while when $b = c = d = 0$, these numbers are simply real numbers. Although it is commonplace for binary operations in algebraic structures to be commutative, such is not the case for products of quaternions. For example, $i \cdot j = k$ but $j \cdot i = -k$. Since De Morgan had shown an interest in Hamilton's research on quaternions as well as other subjects Hamilton had studied, it is likely that De Morgan expected an enthusiastic reply to his letter to Hamilton. Such was not the case, however. Indeed, on 26 October 1852, Hamilton gave a quick but unexpected response:

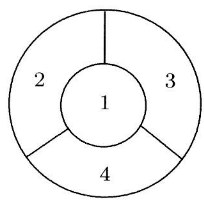
> *I am not likely to attempt your "quaternion" of colours very soon.*

Hamilton's response did nothing however to diminish De Morgan's interest in the Four Color Problem.

Since De Morgan's letter to Hamilton did not mention Frederick Guthrie by name, there may be reason to question whether Frederick was in fact the student to whom De Morgan was referring and that it was Frederick's older brother Francis who was the originator of the Four Color Problem.

In 1852 Frederick Guthrie was a teenager. He would go on to become a distinguished physics professor and founder of the Physical Society in London. An area that he studied is the science of thermionic emission – first reported by Frederick Guthrie in 1873. He discovered that a red-hot iron sphere with a positive charge would lose its charge. This effect was rediscovered by the famous American inventor Thomas Edison early in 1880. It was during 1880 (only six years before Frederick died) that Frederick wrote:

> *Some thirty years ago, when I was attending Professor De Morgan's class, my brother, Francis Guthrie, who had recently ceased to attend then (and who is now professor of mathematics at the South African University, Cape Town), showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof, but the critical diagram was as in the margin.*



> *With my brother's permission I submitted the theorem to Professor De Morgan, who expressed himself very pleased with it; accepted it as new; and, as I am informed by those who subsequently attended his classes, was in the habit of acknowledging where he had got his information.*
>
> *If I remember rightly, the proof which my brother gave did not seem altogether satisfactory to himself; but I must refer to him those interested in the subject. ....*

The first statement in print of the Four Color Problem evidently occurred in an anonymous review written in the 14 April 1860 issue of the literary journal *Athenaeum*. Although the author of the review was not identified, De Morgan was quite clearly the writer. This review led to the Four Color Problem becoming known in the United States.

The Four Color Problem came to the attention of the American mathematician Charles Sanders Peirce (1839–1914), who found an example of a map drawn on a torus (a donut-shaped surface) that required six colors. (As we will see in Chapter 5, there is an example of a map drawn on a torus that requires seven colors.) Peirce expressed great interest in the Four Color Problem. In fact, he visited De Morgan in 1870, who by that time was experiencing poor health. Indeed, De Morgan died the following year. Not only had De Morgan made little progress towards a solution of the Four Color Problem at the time of his death, overall interest in this problem had faded. While Peirce continued to attempt to solve the problem, De Morgan's British acquaintances appeared to pay little attention to the problem – with at least one notable exception.