Topological Groups and Related Structures

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Atlantis Studies in Mathematics
Editor: J. van Mill

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ATLANTIS PRESS
AMSTERDAM – PARIS

World Scientific

Atlantis Press
29 avenue Laumière
75019 Paris, France

For information on all Atlantis Press publications, visit our website at: www.atlantis-press.com.

ISBN: 978-90-78677-06-2
ISSN: 1875-7634

ATLANTIS STUDIES IN MATHEMATICS
VOLUME 1

SERIES EDITOR: **J. VAN MILL**

Atlantis Studies in Mathematics

Series Editor:

J. van Mill, VU University Amsterdam, the Netherlands

(ISSN: 1875-7634)

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Foreword

Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Much of topology is devoted to handling infinite sets and infinity itself; the methods developed are qualitative and, in a certain sense, irrational. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. Very often, the methods here are finitistic in nature.

Because of this difference in nature, algebra and topology have a strong tendency to develop independently, not in direct contact with each other. However, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, and others, topology and algebra come in contact most naturally. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields, transformation groups, topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together; this is the case when they are both determined by the nature of the elements of the set considered (a group of transformations is a typical example). The rules that describe the relationship between a topology and an algebraic operation are almost always transparent and natural — the operation has to be continuous, jointly or separately. However, the methods of study developed in algebra and in topology do not blend so easily, and that is why at present there are very few systematic books on topological algebra, probably, none which can be qualified as a reasonably complete textbook for graduate students and a source of references for experts. The need for such a book is all the greater since the last half of the twentieth century has witnessed vigorous research on many topics in topological algebra. Especially strong progress has been made in the theory of topological groups, going well beyond the limits of the class of locally compact groups. The excellent book [**236**] by E. Hewitt and K. A. Ross just sketches some lines of investigation in this direction in a short introductory chapter dedicated to topological groups.

In the 20th century and during the last seven years many topologists and algebraists have contributed to Topological Algebra. Some outstanding mathematicians were involved, among them J. Dieudonné, L. S. Pontryagin, A. Weil, and H. Weyl. The ideas, concepts, and constructions that arise when topology and algebra come into contact are so rich, so versatile, that it has been impossible to include all of them in a single book; we have made our choice. What we have covered here well may be called "topological aspects of topological algebra". This domain can be characterized as the study of connections between topological properties in the presence of an algebraic structure properly related to the topology.

A. D. Alexandroff, N. Bourbaki, M. I. Graev, S. Kakutani, E. van Kampen, A. N. Kolmogorov, A. A. Markov, and L. S. Pontryagin were among the first contributors to the theory of topological groups. Among those who contributed greatly to this field are W. W. Comfort, M. M. Choban, E. van Douwen, V. I. Malykhin, J. van Mill, and B. A. Pasynkov. These mathematicians have also contributed greatly to other aspects of topological algebra and of general topology.

Though the theory of topological groups is a core subject of topological algebra, a considerable attention has been given to the development of the theory of universal topological algebras, where topology and most general algebraic operations are blended together. This subject started to gain momentum with the works of A. I. Mal'tsev and in later years, some aspects of Mal'tsev's work especially close to general topology were developed by M. M. Choban and V. V. Uspenskij.

The fundamental topic of various types of continuity of algebraic operations was developed in the works of A. Bouziad, R. Ellis, D. Montgomery, I. Namioka, J. Troallic, and L. Zippin. The recent excellent book [**241**] of N. Hindman and D. Strauss contains a wealth of material on algebraic operations on compacta satisfying weak continuity requirements.

One of the leading topics in the general theory of topological groups was that of free topological groups of Tychonoff spaces. It is well represented in our book. A. A. Markov, S. Kakutani, T. Nakayama, and M. I. Graev were at the origins of this chapter. In later years S. A. Morris, P. Nickolas, O. G. Okunev, V. G. Pestov, O. Sipacheva, K. Yamada, and some other mathematicians have worked very successfully in this field.

Our book also contains a very brief introduction to topological dynamics. Among the first who worked in this field were W. Gottshalk, V. V. Niemytzki, and R. Ellis. J. de Vries, one of the later contributors to the subject, wrote the basic monograph [**530**] which gave a strong impuls to its further development. Some recent successes in the field are connected with the names of S. A. Antonyan, V. G. Pestov, S. Glasner, M. Megrelishvili, and V. V. Uspenskij. Well-written, very informative surveys [**379, 378**] by Pestov will orient the reader on this topic.

In this book we refer also to the works of many other excellent mathematicians, among them O. Alas, T. Banakh, L. G. Brown, R. Z. Buzyakova, D. Dikranjan, S. García-Ferreira, P. Gartside, I. I. Guran, K. P. Hart, S. Hernández, G. Itzkowitz, P. Kenderov, K. Kunen, W. B. Moors, P. Nyikos, E. Martín-Peinador, I. Prodanov, I. V. Protasov, D. A. Raĭkov, E. A. Reznichenko, D. Robbie, M. Sanchis, D. B. Shakhmatov, A. Shibakov, L. Stoyanov, A. Tomita, F. J. Trigos-Arrieta, N. Ya. Vilenkin, S. Watson, and E. Zelenyuk. We should also mention that the development of topological algebra was strongly influenced by survey papers [**109, 110, 113**] of W. W. Comfort (and coauthors), and by the books [**410**], [**249**] of W. Roelke and S. Dierolf and of T. Hussain, respectively.

We do not mention here the names of those who have worked recently in the theory of locally compact topological groups. This vast subject is mostly beyond the scope of this book, we have provided only a brief introduction to it.

This book is devoted to that area of topological algebra which studies the influence of algebraic structures on topologies that properly fit the structures. This domain could be called *"Topological invariants under algebraic boundary conditions"*. The book is by no means complete, since this area of mathematics is now rapidly developing in many directions. The central theme in the book is that of general (not necessarily locally compact) topological groups. However, we do not restrict ourselves to this main topic; on the contrary, we try to use it as a starting point in the investigation of more general objects, such as semitopological groups or paratopological groups, for example.

While not striving for completeness, we have made an attempt to provide a representative sample of some old and of some relatively recent results on general topological groups, not restricting ourselves just to two or three topics. The areas covered to a lesser or greater extent are cardinal invariants in topological algebra, separate and joint continuity of group operations, extremally disconnected and related topologies on groups, free topological groups, the Raĭkov completion of topological groups, Bohr topologies, and duality theory for compact Abelian groups.

One of the generic questions in topological algebra is how the relationship between topological properties depends on the underlying algebraic structure. Clearly, the answer to this should strongly depend on the way the algebraic structure is related to the topology. The weaker the restrictions on the connection between topology and algebraic structure are, the larger is the class of objects entering the theory. Because of that, even when our main interest is in topological groups, it is natural to consider more general objects with a less rigid connection between topology and algebra. Examples we encounter in such a theory help us to better understand and appreciate the fruits of the theory of topological groups.

Chapter 1 is of course, of an introductory nature. We define, apart from topological groups, the main objects of topological algebra such as semitopological groups, quasitopological groups, paratopological groups, and present the most elementary and natural examples and the most general facts. Some of these facts are non-trivial, even though they are easy to prove. For example, we establish that every open subgroup of a topological group is closed, and that every discrete subgroup of a pseudocompact group is finite. It is proved in this chapter that every infinite Abelian group admits a non-discrete Hausdorff topological group topology. Quotients, products, and Σ-products are also discussed in Chapter 1, as well as the natural uniformities on topological groups and their quotients.

In the course of the book, we introduce and study several important classes of topological groups. In particular, in Chapter 3 we study systematically ω-narrow topological groups which can be characterized as topological subgroups of arbitrary topological products of second-countable topological groups. An elementary introduction to the theory of locally compact groups is also given in Chapter 3. Then this topic is developed in Chapter 9, where an introduction into the theory of characters of compact and locally compact Abelian groups is to be found. Since there are several good sources covering this subject (such as **[236]**, **[243]**, and **[327]** just to mention a few), we do not pursue this topic very far. However, elements of the Pontryagin–van Kampen duality theory are presented, and the exposition is elementary and practically self-contained.

The celebrated theorem of Ivanovskij and Kuz'minov on the dyadicity of compact groups is proved in Chapter 4. Again, the proof is elementary (though not simple), polished, and self-contained. We apply Pontryagin–van Kampen duality theory to continue the study of the algebraic and topological structures of compact Abelian groups in Chapter 9. The book **[243]** by K.-H. Hofmann and S. A. Morris provides those readers who are interested in the duality theory with considerably more advanced material in this direction.

In Chapter 4 we consider the class of extremally disconnected groups, the class of Čech-complete groups, as well as the classes of feathered groups and P-groups. For each of these classes of groups we prove original and delicate theorems and then establish non-trivial relations between them. Feathered topological groups (called also p-groups) present a natural generalization of locally compact groups and of metrizable groups, that makes them especially interesting.

One of the unifying themes of this book is that of completions and completeness. One can look at completeness in topological algebra either from a purely topological point of view or from the point of view of the theory of uniform spaces; this latter takes into account the algebraic structure much better than the purely topological one. The basic construction of the Raĭkov completion of an arbitrary topological group is presented in Chapter 3; later on, it has many applications. Čech-completeness of topological groups is studied in Chapter 4, and the relationship of Dieudonné completion of a topological group with the group structure is a subject of a rather deep investigation in Chapter 6. In particular, we learn in Chapter 6 that under very general assumptions it is possible to extend continuously the group operations from a topological group to its Dieudonné completion. We also establish that this is not always possible. The class of Moscow groups is instrumental in the theory developed in Chapter 6. The class of $\mathbb{R}$-factorizable groups is studied in Chapter 8. This class serves as a bridge from general topological groups to second-countable groups via continuous real-valued functions. It also turns out to be important in the study of completions of topological groups.

Chapter 5 is devoted to cardinal invariants of topological groups. Invariants of this kind (which associate with topological spaces cardinal numbers "measuring" the space under consideration in one sense or another), play an especially important role in general topology; probably, this happens because the techniques they provide fit best the set-theoretic nature of general topology. So one may expect that in the study of non-compact topological groups cardinal invariants should occupy a prominent place. The following phenomenon makes the situation even more interesting: The structure of topological groups turns out to be much more sensitive to restrictions in terms of cardinal invariants than the structure of general topological spaces. For example, metrizability of a topological group depends only on whether the group is first-countable or not (Birkhoff–Kakutani's theorem). For paratopological groups the statement is no longer true (the Sorgenfrey line witnesses this); however, a weaker theorem holds: every first-countable paratopological group has a G_δ-diagonal. How delicate problems involving cardinal invariants of compact groups can be, is shown by the following simple result: It is not possible either to prove, or to disprove in ZFC that every compact group of cardinality not greater than $\mathfrak{c} = 2^\omega$ is metrizable.

In Chapter 7, a very powerful and delicate construction is presented — that of a free topological group over a Tychonoff space. Under this construction, any Tychonoff space X can be represented as a closed subspace of a topological group $F(X)$ in such a way that every continuous mapping of a space X into a space Y can be uniquely extended to a continuous homomorphism of $F(X)$ to $F(Y)$. The set X, of course, serves as an algebraic basis of $F(X)$. However, the relationship between the topology of X and that of $F(X)$ is the most intriguing; there are many unexpected and subtle results on free topological groups and quite a few unsolved problems. One of the most important theorems here states that the cellularity of the free topological group $F(X)$ of an arbitrary compact Hausdorff space X is countable. Curiously, one can demonstrate that this happens not because of the existence of a regular measure on $F(X)$ (in contrast with the case of compact groups). In fact, such a measure on $F(X)$ exists if and only if X is discrete.

Yet another major topic in the book is that of transformation groups, and the closely associated concepts of homogeneous spaces and of groups of homeomorphisms. A section is devoted to these matters in Chapter 3, where it is established that the group of isometries of a metric space is a topological group, when endowed with the topology of pointwise convergence. It is proved in this connection that every topological group is topologically isomorphic to a subgroup of the group of isometries of some metric space. This provides an important technical tool for some arguments.

Frequently, results on topological groups are followed by a discussion of other structures of topological algebra, such as semitopological and paratopological groups. This is done in almost every chapter. However, we have also devoted the whole of Chapter 2 to basic facts regarding such objects. A *paratopological group* is a group G with a topology such that the multiplication mapping of $G \times G$ to G is jointly continuous. A *semitopological group* G is a group G with a topology such that the multiplication mapping of $G \times G$ to G is separately continuous. A *quasitopological group* G is a group G with a topology such that the multiplication mapping of $G \times G$ to G is separately continuous and the inverse mapping of G onto itself is continuous. A natural example of a paratopological group is obtained by taking the group of homeomorphisms of a dense-in-itself locally compact zero-dimensional non-compact space, with the compact-open topology. The Sorgenfrey line under the usual addition is a paratopological group which is hereditarily separable, hereditarily Lindelöf and has the Baire property.

In 1936, D. Montgomery proved that every semitopological group metrizable by a complete metric is, in fact, a paratopological group. In 1957, R. Ellis showed that every locally compact semitopological group is a topological group. In 1960, W. Zelazko established that each completely metrizable semitopological group is a topological group. Later, in 1982, N. Brand proved that every Čech-complete paratopological group is a topological group. Recently A. Bouziad made a decisive contribution to this topic. He proved that every Čech-complete semitopological group is a topological group. This theorem naturally covers and unifies both principal cases, those of locally compact semitopological groups and of completely metrizable semitopological groups.

Since each Čech-complete topological group is paracompact, Bouziad's theorem implies that every Čech-complete semitopological group is paracompact. These and related results, with applications, are presented in Chapter 2. In this same chapter we construct an operation of a rather general nature on the Čech–Stone compactification βG of an arbitrary discrete group G. With this operation, the compact space βG becomes a right topological group. This structure has interesting applications; we mention some of them in problem sections. The reader who wants to learn more on this subject is advised to study the recent book [**241**] by N. Hindman and D. Strauss.

We formulate and discuss quite a few open problems, many of them are new. Each section is followed by a list of exercises and problems, including open problems. However, we should warn the reader that some of the new open problems might turn out not to be difficult after all. That does not necessarily mean that they should have been discarded. The main interest of many of the new questions we have posed lies in the fact that they delineate a new direction of research. On many occasions exercises and problems are provided with hints, references, and comments. In this way, many additional directions and topics are introduced. Here are two outstanding examples of old unsolved questions. Is it possible to construct in ZFC a non-discrete extremally disconnected topological group? Is it possible to construct in ZFC a countable non-metrizable topological group G such that G is a Fréchet–Urysohn space?

We would be grateful for the information on the progress of open problems posed in this book.

We hope that this book will achieve several goals. First, we believe that it can be used as a reasonably complete introduction to the theory of general topological groups beyond the limits of the class of locally compact groups. Second, we expect that it may lead advanced students to the very boundaries of modern topological algebra, providing them with goals and with powerful techniques (and maybe, with inspiration!). The exercise and problem sections can be especially useful in that respect. One can use this book in a research seminar on topological algebra (with an eye to unsolved problems) and also in advanced courses — at least four special courses can be arranged on the basis of this book. Fourth, we expect that the book will serve quite effectively as a reference, and will be helpful to mathematicians working in other domains of mathematics.

The standard reference book for general topology is R. Engelking's book *General Topology* [**165**]. We expect that the reader either knows the basic facts from general topology that we need, or that he/she will not find it too difficult to extract the corresponding information from [**165**].

We wish to express our deep gratitude to the second author's former students Constancio Hernández García and Yolanda Torres Falcón for their continued help in our work on this book over many years. We are also indebted to Richard G. Wilson whose comments enabled us to improve the text.

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February 21, 2008

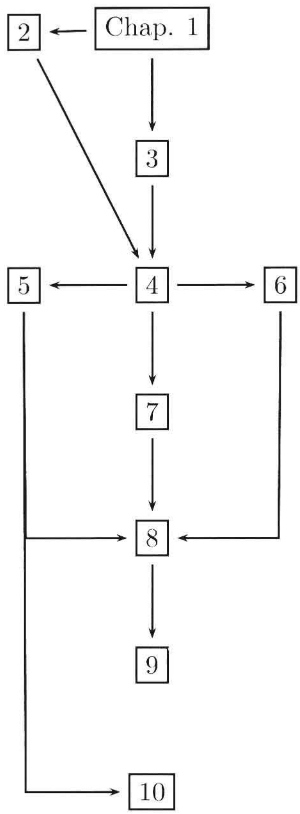

FIGURE 1. Logical dependence of chapters.

Contents

Chapter 1

Introduction to Topological Groups and Semigroups

Notation. We write $\mathbb{N}$ for the set of positive integers, ω for the set of non-negative integers, and $\mathbb{P}$ for the set of prime numbers. The set of all integers is denoted by $\mathbb{Z}$, the set of all real numbers is $\mathbb{R}$, and $\mathbb{Q}$ stands for the set of all rational numbers.

The symbols τ, λ, κ are used to denote infinite cardinal numbers. A cardinal number τ is also interpreted as the smallest ordinal number of cardinality τ. Each ordinal is the set of all smaller ordinals. Thus, ω is both the smallest infinite ordinal number and the smallest infinite cardinal number.

All topologies considered below are assumed to satisfy T_1-separation axiom, that is, we declare all one-point sets to be closed. The closed unit interval $[0, 1]$ of the real line $\mathbb{R}$, with its usual topology, is denoted by I, and Sq stands for the convergent sequence $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ with its limit point 0, also taken with the usual topology. We use the symbol $\mathbb{C}$ to denote the complex plane with the usual sum and product operations, while $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle with center at the origin of $\mathbb{C}$.

1.1. Some algebraic concepts

In this section we establish the terminology and notation that will be used throughout the book.

In dealing with groups, we will adhere to the multiplicative notation for the binary group operation. In discussions involving a multiplicative group G, the symbol e will be reserved for the identity element of G.

We are very much concerned with groups in this course. For many purposes, however, it is natural and convenient considering semigroups. A *semigroup* is a non-void set S together with a mapping $(x, y) \to xy$ of $S \times S$ to S such that $x(yz) = (xy)z$ for all x, y, z in S. That is, a semigroup is a non-void set with an associative multiplication. Given an element x of a semigroup S, one inductively defines

$$x^2 = xx,\ x^3 = xx^2,\ \ldots,\ x^{n+1} = xx^n,$$

for every $n \in \mathbb{N}$. The associativity of multiplication in S implies the equality $x^n x^m = x^{n+m}$ for all $x \in S$ and $n, m \in \mathbb{N}$.

An element e of a semigroup S is called an *identity* for S if $ex = x = xe$ for every $x \in S$. Not every semigroup has an identity (see items 4) and 6) of Example 1.1.1). However, if a semigroup S has an identity, then it is easy to see that this identity is unique. Whenever we use the symbol e without explanation, it always stands for the identity of the semigroup under consideration.

A semigroup with identity is called *monoid*. An element a of a monoid M is said to be *invertible* if there exists an element b of M such that $ab = e = ba$. Note that if a is an invertible element of a monoid M, then the element $b \in M$ such that $ab = e = ba$ is unique. Indeed, suppose that $ab = e$, $ba = e$, $ac = e$, and $ca = e$. Then we have

$$c = ec = (ba)c = b(ac) = be = b.$$

This fact enables us to use notation a^{-1} for such an element b of M. We also say that b is the *inverse* of a. It is clear that $(a^{-1})^{-1} = a$ for each invertible element $a \in M$. Further, one can define negative powers of an invertible element $a \in M$ by the rule $a^{-n} = (a^{-1})^n$, for each $n \in \mathbb{N}$. It is a common convention to put $a^0 = e$. We leave to the reader a simple verification of the equality $a^n a^m = a^{n+m}$ which holds for all $n, m \in \mathbb{Z}$.

If every element a of a monoid M is invertible, then M is called a *group*.

Let S be a semigroup. For a fixed element $a \in S$, the mappings $x \mapsto ax$ and $x \mapsto xa$ of S to itself are called the *left* and *right actions* of a on S, and are denoted by λ_a and ϱ_a, respectively.

If G is a group, the mapping $x \mapsto x^{-1}$ of G onto itself is called *inversion*. Left and right actions of every element $a \in G$ on G are, in this case, bijections. They are called *left* and *right translations* of G by a.

EXAMPLE 1.1.1. Each of the following is a semigroup but not a group.

1) The set $\mathbb{Z}$ of all integers with the usual multiplication.
2) The set $\mathbb{Q}$ of all rational numbers with the usual multiplication.
3) The set $\mathbb{R}$ of all real numbers with the usual multiplication.
4) The set of all positive real numbers with the usual addition in the role of the product operation.
5) The set $\mathbb{N}$, in which the product of x and y is defined as $\max\{x, y\}$.
6) The set $\mathbb{N}$, in which the product of x and y is defined as $\min\{x, y\}$.
7) Any set S with $|S| > 1$, where the product xy is defined as y.
8) Any set S with $|S| > 1$, where the product xy is defined as x.
9) The set $S(X, X)$ of all mappings of a set X to itself with the composition of mappings in the role of multiplication, where $|X| > 1$. □

In items 4) and 6) of the above example, the corresponding semigroups have no identity. The semigroups in 1)–3), 5), and 9) are monoids.

Now we present a few standard examples of groups.

EXAMPLE 1.1.2. Each of the following is a group:

1) The set $\mathbb{Z}$ of all integers with the usual addition in the role of multiplication.
2) The set $\mathbb{Q} \setminus \{0\}$ of all non-zero rational numbers with the usual multiplication.
3) The set $\mathbb{R} \setminus \{0\}$ of all non-zero real numbers with the usual multiplication.
4) The set of all positive real numbers with the usual multiplication.
5) The set $\{0, 1\}$ with the binary operation defined as follows:

$$0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0.$$

This group is denoted by $\mathbb{Z}(2)$ or by D; it is called the *cyclic group* of two elements, or the *two-element group*. More generally, for an integer $n > 1$, let $\mathbb{Z}(n) = \{0, 1, \ldots, n-1\}$