Intermediate Algebra

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Preface

This text has been designed to serve two groups of students. For those who have had little or no algebra, the first eight chapters might be used as a beginning course. For those students who have completed one year of high school algebra and those who may have completed two years in high school but need an intensive review before continuing with more advanced courses, the text is well adapted. In such case, a minimum review could be provided with less emphasis on Chapters 2, 3, 4, 5, and 7 and a more intensive coverage of the remaining chapters.

The axiomatic approach has been used in an attempt to provide the student with a thorough understanding of the meaning and nature of mathematics. Extensive emphasis is given to logical justification for each integral part of the development of the course; this is followed by illustrative examples of problems associated with new concepts. In introducing new topics, an attempt has been made to present clearly and completely the necessary theory in a manner that will not leave the student in a state of confusion.

Early chapters offer an intensive review of the basic principles of elementary algebra with the additional topics that are normally included in a second course in algebra. The remaining chapters provide the necessary training to prepare the student to continue with advanced courses in mathematics, as well as in the scientific and technical fields.

The text is adaptable to diversified groups and various classroom conditions. For the student who expects to continue the study of mathematics, there is rigorous development of the structure of algebra consistent with the level of the student that gives a solid foundation for further study. For those students who are following training programs in the various fields of technology, there is the same provision for basic foundations as well as many illustrative problems associated with the theoretical development and extensive drill exercises to assist in the mastery of mathematical skills. This course provides the necessary mathematical prerequisite to the study of elementary statistics that is rapidly becoming a requirement in many curricula. In all fields of scientific study and the various engineering areas, this book provides adequate emphasis for those students to find it necessary to strengthen their basic mathematical understanding.

Extensive drill in factoring is provided to enable the student to master the technique of that phase of algebra and readily make use of its applications throughout the course.

Sets have been used throughout the text to give meaningful significance to developing algebra as a mathematical system.

Considerable attention has been given to the graphing of relationships. The traditional graphing of equalities is included, although much emphasis is placed on the graphing of inequalities.

Drill exercises are arranged in order of increasing difficulty thereby presenting a minimum course through partial coverage and additional exercises for the more enterprising student. Certain topics, evolved late in the chapter, may be omitted without destroying the continuity of the course.

Answers to the odd-numbered problems are included in the text. Answers to the even-numbered problems are available for instructors.

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D. S. R.

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1. INTRODUCTION

A course in intermediate algebra presupposes that the student has completed some formal study of algebra. In this course it is desirable to duplicate some of the previous concepts covered. This duplication will be minimized but used sufficiently to aid the student who is experiencing a time lapse since his previous training in algebra.

To develop a thorough understanding of algebra at this level, it will be our concern to learn *why* as well as *how* we do the necessary operations. In any discussion it is mandatory that certain undefined terms be used. In this discussion we shall use only those undefined terms with which the student is familiar. This will consist of such terms as *point, line, circle, addition*, and *multiplication*.

With the use of those undefined terms with which it is assumed students are familiar, we state a set of laws, axioms, or postulates that we assume those undefined terms obey. These laws, axioms, or postulates, then, are not defined but merely *assumed* to be true. The laws constitute a set of properties of the numbers with which we shall be concerned.

Throughout the development of our course in algebra, names for new concepts, symbols, and mathematical operations will, of necessity, have to be assigned. As these new names or terms emerge they will be defined.

As we continue to enlarge and build a volume of knowledge about our system of mathematics, a great many new ideas must be developed.

Some of these new ideas will be statements of fact, called theorems. A *theorem* is a proposition that can be proved to be true through the use of a series of logical statements, each of whose validity is based on the laws or axioms or a definition. Once a theorem has been proved it, too, can be used as a valid reason for a logical step in the proof of another theorem.

In summary, then, our system of mathematics is developed through the use of undefined terms, defined terms, a set of axioms or laws, and theorems.

2. REAL NUMBERS

Natural numbers are those numbers that are used for counting. Thus, the numbers 1, 2, 3, $\cdots$ (the three dots mean *and so on indefinitely*) are natural numbers.

Zero was used originally, in all probability, as a place holder. To distinguish the number 301 from 31 it was necessary to introduce some symbol to hold the tens place and at the same time to indicate that there was no value in tens place.

As civilization became more complex and as man began to examine number concepts and operations with numbers, it was necessary to introduce certain new types of numbers. If the natural numbers were used to represent the profit made by traders, there eventually arose a need for some type of number to represent any losses resulting in their transactions. Thus, negative numbers may have been invented to represent such concepts. With the introduction of the sciences, the men associated with the development of those branches of study found additional uses for negative numbers. If the natural numbers were used to represent temperature readings above zero, the negative numbers provided a logical pattern to represent any reading below zero. In a similar manner, if the natural numbers were used to denote distances above sea level, the negative numbers made available an easy and reasonable system whereby distances below sea level could be recorded mathematically.

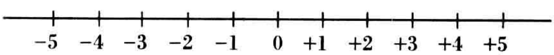

Figure 1

All these numbers can be shown on the number line in Figure 1. Numbers to the right of zero on that line are positive numbers and those to the left of zero are negative numbers.

The numbers shown in Figure 1 are whole numbers or *integers*. Eventually the integers did not provide a set of sufficient number symbols to fulfill the needs of an expanding civilization. As the numerous transactions between members of society became more complicated and the enquiring mind delved deeper into the operations with numbers, there arose the need for number symbols that represented parts of integers. This exigency led to the introduction of both common and decimal fractions. At the present time we shall call any number in the form a/b, where a and b are integers (except b cannot equal zero), a *common fraction*. (The exclusion of division by zero will be explained later in the text.) The common fraction $\frac{2}{3}$ means that we are using two of three equal parts of something. The number $2\frac{1}{2}$ means two whole parts of something and an additional one-half of another whole part of the same thing. Thus $2\frac{1}{2}$ miles denotes two whole miles and one-half of one more mile. Decimal fractions are those of the form 0.5, 1.25, or 9.96, as were used in your study of arithmetic.

Common and decimal fractions may be located on the number line shown in Figure 1. The number $\frac{1}{2}$ or its decimal equivalent 0.5 would be located midway between zero and $+1$. Likewise, $-\frac{1}{2}$ or -0.5 would be located at a point midway between zero and -1. In a similar manner, $3\frac{1}{2}$ would be a point on the number line midway between $+3$ and $+4$, and $-3\frac{1}{2}$ would be a point midway between -3 and -4.

The positive and negative integers, zero, the positive and negative common fractions, and the positive and negative decimal fractions constitute what we call our *real number system.*

3. PROPERTIES OF REAL NUMBERS

The following laws apply to our real number system.

Law 1. ***The law of closure for addition.*** *If a and b are any real numbers, there exists a unique (one and only one) real number called the sum of a and b. The sum of a and b is indicated by $a + b$.*

This law states that the sum of any two real numbers is also a real number or that the system is closed under addition.

Law 2. ***The law of closure for multiplication.*** *If a and b are any real numbers, there is a unique real number called their product. The product of a and b may be indicated by $a \cdot b$, or $(a)(b)$, or just ab.*

The law of closure for multiplication tells us that the product of two real numbers is also a real number.

Law 3. ***The commutative law for addition.*** *If a and b are any real numbers, then $a + b = b + a$.*

The commutative law for addition states that the sum of any two real numbers remains the same regardless of the order in which the numbers are added. This is consistent with the practice of checking the addition of columns of numbers upward after having added the columns from top to bottom.

Law 4. ***The commutative law for multiplication.*** *If a and b are any real numbers, then $ab = ba$.*

The commutative law for multiplication means that the numbers to be multiplied may be multiplied in any order. Thus, $2 \cdot 3 = 6$ and $3 \cdot 2 = 6$.

Law 5. ***The associative law for addition.*** *If a, b, and c are any real numbers, then $a + (b + c) = (a + b) + c$.*

The associative law for addition means that the addends (the numbers to be added) may be associated or grouped together in any manner. In other words, we add the sum of b and c to a or add c to the sum of a and b. For example, $2 + (5 + 7) = (2 + 5) + 7$.

$$\begin{aligned} 2 + (5 + 7) &= (2 + 5) + 7 \\ 2 + 12 &= 7 + 7 \\ 14 &= 14 \end{aligned}$$

Law 6. ***The associative law for multiplication.*** *If a, b, and c are any real numbers, then $a(bc) = (ab)c$.*

The associative law for multiplication means that a product remains the same regardless of the order in which the factors are associated or grouped together. We may find the product of b and c and multiply that product by a or find the product of a and b and multiply that product by c. Thus, $2(3 \cdot 5) = (2 \cdot 3)5$.

$$\begin{aligned} 2(3 \cdot 5) &= (2 \cdot 3)5 \\ 2(15) &= (6)5 \\ 30 &= 30 \end{aligned}$$

Law 7. ***The distributive law for multiplication with respect to addition.*** *If a, b, and c are any real numbers, then $a(b + c) = ab + ac$.*

The distributive law for multiplication with respect to addition means that the factor a is distributed over all the addends included within the parentheses. Thus, $2(4 + 3) = 2 \cdot 4 + 2 \cdot 3 = 8 + 6 = 14$. This problem could be worked by first finding the sum of the addends within the parentheses and then multiplying that sum by 2. Thus $2(4 + 3) = 2(7) = 14$.

Law 8. ***If a is any real number, there exists a unique number 1 such that $a \cdot 1 = a$.*** *The number by which a, representing any real number, may be multiplied to obtain a product equal to a is called the multiplicative identity.* Thus, the multiplicative identity is 1.

Law 9. ***If a is any real number, there exists a unique number 0 such that $a + 0 = a$.*** *The number that may be added to a to obtain a sum which is equal to a is known as the additive identity.* Thus, the additive identity is 0.

Law 10. ***If a is any real number, there exists a unique real number $-a$ such that $a + (-a) = 0$.*** This law tells us that the sum of any real number and its negative counterpart is always 0.

Law 11. ***If a is any real number except 0, there exists a unique real number such that $a(1/a) = 1$.*** One divided by any given number is called the reciprocal of that number. Thus, any real number multiplied by its reciprocal is always equal to 1.

Law 12. ***The reflexive relationship.*** *If a is any real number then $a = a$.*

Law 13. ***The law of symmetry.*** *If a and b are any real numbers and if $a = b$, then $b = a$.*

Law 14. ***The transitive relationship.*** *If a, b, and c are any real numbers and $a = b$ and $b = c$, then $a = c$.* This law states that quantities equal to the same or equal quantities are equal to each other.

Law 15. ***If a, b, x, and y are any real numbers and $a = b$ and $x = y$, then $a + x = b + y$.*** This law states that, if equals are added to equals, the sums are equal.

Law 16. ***If a, b, x, and y are any real numbers and $a = b$ and $x = y$, then $ax = by$.*** This law states that if equals are multiplied by equals the products are equal.

If $a = b$, we have a statement of equality. Such a statement is called an *equation.* An equation has two parts, one on each side of the equals symbol. That part on the left of the symbol is said to be the left member of the equation and that part on the right side the right member of the equation. The symbol $\neq$ means that the two parts of the statement are not equal. Thus $a \neq b$ means that a is not equal to b. The symbol gives no clue as to the relative values of a and b, only that they are unequal. The symbol $<$ means *less than* and the symbol $\leq$ means *less than or equal to*. Thus, $a < b$ is read: "a is less than b." Likewise, $a \leq b$ is read: "a is less than or equal to b." If the symbol is reversed as in the expression $a > b$, we read it: "a is greater than b." In like manner, the statement $a \geq b$ is read: "a is greater than or equal to b."

Law 17. ***If a and b are any real numbers, either $a = b$, $a < b$, or $a > b$.*** This is called the law of *trichotomy*.

If $a = a$, the relationship is said to be *reflexive*.

Laws 14, 15, and 16 are sometimes called axioms. An *axiom* is defined as a basic assumption.

Certain of the laws just given may be extended such that they are more inclusive than the original statement might imply. Law 3, the commutative law for addition, holds true regardless of how many numbers are to be added. Thus, $a + b + c + d = a + b + d + c = d + a + b + c$, etc. In a similar manner, Law 4 (the commutative law for multiplication) applies to the use of any number of multipliers. Hence, $abcd = adbc = bcad$, etc.

The distributive law, also, may be extended so that it is much more comprehensive. As stated previously, $a(b + c) = ab + ac$. This is called the *left* distributive law. Since $a(b + c)$ implies multiplication, we can use the commutative law for multiplication and write the expression $(b + c)a$. Again, the a may be distributed over $(b + c)$ such that $(b + c)a = ab + ac$, and is referred to as the *right* distributive law.

If a and b represent real numbers, by the law of closure, $(a + b)$ is a real number. Then, to keep our system consistent, we must be able to distribute such a number over the various terms in another multiplier. Thus we extend the distributive law to include the following:

$$\begin{aligned}(a + b)(x + y) &= (a + b)x + (a + b)y &&\text{(by the left distributive law)}\\ &= ax + bx + ay + by &&\text{(by the right distributive law)}\end{aligned}$$

In a similar manner,

$$\begin{aligned}(a + b + c)(x + y + z) &= (a + b + c)x + (a + b + c)y + (a + b + c)z\\ &= ax + bx + cx + ay + by + cy + az + bz + cz\end{aligned}$$

Hence, the distributive law applies regardless of how many terms are given in either of the numbers to be multiplied together.

EXERCISE 1

If all letters used in the following problems represent real numbers, state which of the foregoing laws make the statement true.

1. $x(y + z) = xy + xz$
2. $mnp = pmn$
3. If $x = 3$, then $4x = 12$.
4. $x(y + z + 1) = xy + xz + x$
5. $m + n + p = n + p + m$
6. If $a \neq b$, then either $a < b$ or $a > b$.
7. $c + (d + e) = (c + d) + e$
8. $m + 0 = m$
9. If $a = b$, then $a + d = b + d$.
10. If $m = n$, then $n = m$.
11. If $m = n$ and $x = y$, then $m + x = n + y$.
12. If $p = q$ and $p = r$, then $q = r$.
13. If $m = n$ and $r = s$, then $mr = ns$.
14. $ty = yt$
15. $z \cdot 1 = z$

4. DEFINITIONS

The definitions of this section will be used throughout this book.

A symbol that is used to represent a number is called a *numeral*. In our system of mathematics we generally use the nine single-digit natural numbers and zero singly or in combinations as numerals. The number 5 may be represented by 5 (the Arabic numeral for 5), by V (the Roman numeral), the symbol 𝍸 that we sometimes use when we tally a group of scores, or any other symbol that might be invented for this purpose.

An *even* number is an integer that is exactly divisible by 2. If x represents any number, then $2x$ will always be an even number. An *odd* number is any number that is not divisible by 2 and may be represented by the expression $2x - 1$.

In the expression $4ax + 3ab - 2by$, the $4ax$, the $+3ab$, and the $-2by$ are called the *terms* of the expression. Each member of this series, together with the sign that precedes it, is called a term of the expression. If no sign is shown with a term or no sign is given with the first term in an expression of mathematical terms, it is assumed to be positive. Any expression containing two or more terms is called a *multinomial* and one with a single term is known as a *monomial*. Special cases of the multinomial are an expression of two terms, called a *binomial*, and one with three terms, called a *trinomial*. The expression $x + y$ is a binomial and $3x + y - 2z$ is a trinomial, but either may be referred to as a multinomial.

A *rational* number is any number that can be expressed in the form a/b, where a and b are integers and $b \neq 0$. Rational numbers consist of common fractions and integers, since any integer may be written over 1 (6/1) without changing its value. Thus $\frac{3}{5}$, $-\frac{1}{2}$, and 5 are rational numbers. An *irrational* number is any number that is not rational. In other