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INVARIANT SETS FOR WINDOWS

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INVARIANT SETS FOR WINDOWS
Resonance Structures, Attractors, Fractals and Patterns

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Preface

It has been known since antiquity that Nature is full of wonderful geometric structures (cellular, helical, vertex, crystalline etc), and some remarkable purely mathematical structures have been discovered in the last century (resonance structures, strange attractors, fractals). Besides being of great scientific interest, these structures often possess a very beautiful visual form. In recent years the structures have been the object of an extensive research activity in various fields of Science. The dynamical theory of formogenesis is widely discussed in the literature.

This book attempts to provide both the visual presentation and theoretical analysis of the structures arising in nonlinear dynamical systems. The WInSet program (developed by the authors) is used to generate the images of many invariant structures related to a variety of phenomena of nonlinear dynamics. The images of structures dealt with in the book may be referred to as "fractal design" or "esthetic chaos".

The book consists of two parts. Part I is intended for a wide readership and is concerned with the WInSet software for the visualization of invariant sets of classical nonlinear dynamical systems. The features of the program are described, and the standard (built-in) maps, differential equations and fractals are listed, for which WInSet can draw the invariant sets. The program functions in Windows 95 environment and can be used for computer-aided design.

Part II presents the mathematical investigation of invariant sets in low-dimensional dynamical systems and diffusion equations, and explains how the invariant structures appear and requires a sound mathematical knowledge.

Chapters 1, 4–7 were written by A. Morozov (except Section 4.4.1 and Section 4.4.2 written by T. Dragunov), Chapter 2 — by O. Malysheva and T. Dragunov, Chapter 3 — by S. Boykova, Chapter 8 — by A. Morozov and O. Malysheva. Writing and debugging of the WInSet source code in Delphi-3 was mainly done by T.Dragunov, while some modules were written by the other authors. The compiler for user-defined systems was written by S. Boykova and T. Dragunov.

WInSet grew out of the program called Mader developed by Morozov in early 90-s.

The book is based on the monograph "Invariant Sets of Dynamical Systems for Windows" published in Russia in 1998, written by the same authors [68]. The Chapter 8 has been added especially for the English edition. Besides, Chapters 1–3 have been revised, and the new version of the WInSet application has been created.

The authors are grateful to Mark Shereshevsky for his valuable help in the preparation of the English version.

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Contents

Chapter 1

Introduction

1.1. Invariant Structures Everywhere

What do we mean when we say "invariant structures" or "invariant set"? According to the encyclopedic dictionary the noun "invariant" originates from the latin "invarians" (changeless) and means a value (a quantity) which is left unchanged by certain transformations. If this property is displayed not by a single value, but a set of values (or points), then such a set is referred to as invariant set. Mathematicians give more rigorous definitions using special mathematical concepts. One of such definitions will be given below, when we discuss dynamical systems. A simple example of an invariant set is provided by the collection of figures with a fixed area on the plane under an isometry (or any area-preserving transformation). A square with all four corners positioned on the coordinate axes in the plane (thus centered at the origin) is an invariant set under the rotation by 90^0. This type of invariance underlies the concept of symmetry. Another example of an invariant set is given by the orbit of the Earth under the transformations in the space of the Solar system induced by the planetary motions in time.

The Introduction and Part I of the book are intended for a wide audience and presuppose no special preliminary background. The only exception may be the notion of "dynamical system" which is closely related to the concept of "differential equations". The reader who is not familiar with (or interested in) differential equations may limit his or her reading by our consideration of discrete dynamical systems (i.e. maps) and experiment with the fractals generated by them. In this part of the book we mainly focus on the WInSet computer program and the results obtained by using it. The program not only enables the user to observe the invariant sets of dynamical systems (in particular, the fractals), but also allows one to introduce his own dynamical system and investigate its behavior and the fractals it generates.

Its sofisticated interface leads the user into the wonderful world of fractals, a world full of fascinating colors and harmony.

Part II is devoted to the description of invariant sets and demands from the reader the mathematical background in the amount of the first three years of college math. After having read it you will become familiar with the mathematical nature of the phenomena you observe on the computer screen when using, for example, WInSet software. Behind them are the frontline problems of the nonlinear dynamics, such as the nonlinear resonance, self-oscillations, irregular invariant sets (strange attractors, fractals etc.)

Shortly speaking, the main topic of the book is the visualization of invariant sets, and the demonstration of their beauty and somewhat mysterious inner harmony. We hope that this visual splendor will provide the inspiration (so important in the progress of sciences) for those who have a scientific interest in the nonlinear phenomena, as well as for those who are just curious about all this marvelous pictures.

1.1.1. Resonance Structures in Celestial Mechanics

Celestial mechanics is one of the most fascinating sciences, since it deals with the fundamental laws which govern the universe. It may even be considered the cradle of the calculus and the theory of differential equations and dynamical systems (for it is the differential equations that describe the motions of celestial bodies). In celestial mechanics the equations are usually assumed to be conservative, i.e. they preserve a certain quantity (the full energy of the system, for example). This is exactly the invariance property. The conservative systems are normally presented in a special form - as a so-called Hamiltonian system. In the modern mechanics and mathematics there is a field called Hamiltonian mechanics which deals with the general properties of this type of systems. The Hamiltonian systems generate resonance structures of astounding beauty (see Chapter 4).

Although a significant progress has been achieved in the study of celestial mechanics, one crucial question remains open. It is the question of the stability of Solar system. Whether the configuration of Solar system will remain stable over an infinite interval of time is not known. So it is unclear whether, for instance, Earth will always travel along an elliptic orbit close to the current one, or it will undergo a significant deformation and our planet will fall on the Sun or, on the contrary, will leave the Solar system altogether. The same question could be posed about the satellites of the planets. The question has been studied for more than two hundred years and is still unanswered. So, it is not known whether the current configuration of the Solar system is a result of a long evolutionary process or it "was born like this". The evolutionary model seems more plausible. In favor of it speaks the analysis of quasi-conservative systems and, most of all, the resonance structures in such systems.

The phenomena of "capture in a resonance" and "synchronization" can explain the "resonance" structure of the Solar system.

The **macroproblems** of celestial mechanics are similar, in their nature, to the **microproblems** on the structure of atom. Both are the problems on the motion of a "particle" in a central field. In the former problem the "particle" is a planet, in the latter — the electrons.

The resonance structures are invariant ones. Further in the book (Chapters 4-6) we shall discuss these structures in more detail by considering some simplified models. Right now let us look at the resonance equations in the solar system. Currently the scientists know with good precision the mean daily motions of the planets ω_j, $j = 1, ..., 9$ (expressed in degrees). So, one can check if the resonance equations are satisfied.

A resonance is said to occur if the equality

$$\sum_{j=1}^{n} k_j \omega_j = 0,$$

holds, where k_j are integer numbers, and n is the number of frequencies. If all planets are taken into considerations, then $n = 9$.

The resonance condition is the condition of commensurability of the frequencies. The number $||k|| = \sum_{j=1}^{n} |k_j|$ is called the order of the resonance. The main perturbing body in Solar system is Jupiter whose mass is many times larger than the masses of the other planets. So, let us consider the lowest resonances in the three body problem "Sun-Jupiter-planet". The lowest order of resonance $||k|| = 3206$ observed for the system Sun-Jupiter-Neptune is very large [29]. It turns out that the higher is the order of the resonance the less it is manifested in the motion of the bodies. Let us then consider not the exact resonances, but rather those close to the exact ones, i.e. instead of the resonance equality consider the condition in the form of inequality:

$$k_1 \omega_1 + k_2 \omega_{\text{пл}} \le \varepsilon,$$

where $\varepsilon > 0$ is small enough, ω_1 is the mean daily motion of the Jupiter, and ω_{pl} is the mean daily motion of the planet. If, for instance, $\varepsilon = 5''$ (5 seconds), then the lowest order of resonance in the system Sun-Jupiter-Saturn is 7 ($k_1 = 2, k_{pl} = 5$), and the highest one is 649 ($k_1 = 617, k_{pl} = 32$) observed in the system Sun-Jupiter-Venus [29]. For the inner (with respect to Jupiter) planets increasing the "aperture" ε significantly lowers the maximal order of resonance.

Resonance relations are also observed in the satellite systems, as well as in planet-satellite type of systems. Laplace noticed the remarkable triple-frequency resonance in the satellite system of Jupiter formed by the mean motions of Jo, Europe and Ganimedes: $\nu_1 - 3\nu_2 + 2\nu_3 \approx -0.0003°$ (where ν_1, ν_2, ν_3 are the mean motions of Jo,

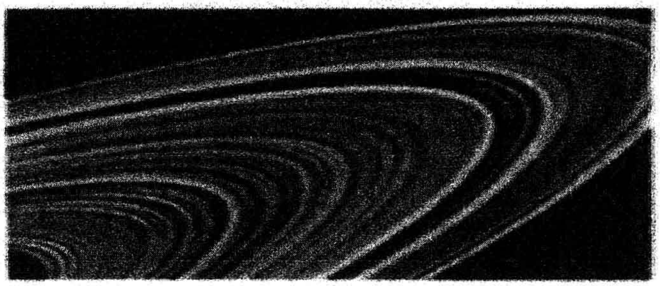

Fig. 1.1. The structure of the rings of Saturn (fragment).

Europe and Ganimedes respectively. The system Earth-Moon has the resonance with $k_1 = k_2 = 1$.

The satellite system and the rings of Saturn is a unique formation. Fig 1.1 shows a fragment of Saturn's system of rings obtained by Voyager-2 in August 1981 (reproduced from [93]). The structure of the rings looks somewhat like a fractal. Such invariant structures will be discussed in Section 1.1.3. and Chapter 7.

Another remarkable formation in the solar system is the belt of small planets (asteroids) which consists of more than 1800 objects. The statistical distribution of asteroids according to their mean motion is extremely irregular. Amazingly, some of its maxima and minima occur at the resonance values of the mean motion in system Jupiter-asteroid [29].

1.1.2. Cellular, Spiral, Vortex and Crystal Structures

Let us turn to more "earth-bound" sciences. In everyday life we meet (usually unaware of it) myriads of fascinating invariant structures of very diverse nature. In this section we briefly discuss some of those beauties. A good reading on various cases of formogenesis (formation of structures) in physics, chemistry, biology etc. is the recent book by Rabinovich and Ezersky [84].

Almost everyone heard that the structure of a honeycomb is hexagonal (the hexagons fit together, side to side, to form a "tiling" of the plane). A structural regularity can be observed in the configurations of flowers, seeds, leaves. The leaves of a shoot of a plant and the seeds in the sunflower display a regular *spiral* structure. There is a tendency towards a spiral in Nature. The botanists call it *philotaxis.*

Perhaps the most striking manifestation of Nature's romance with the spiral is the beautiful coach shell Nautilus pompilius [1].

Following Henry Weyl [104] we now describe one mathematical interpretation of the harmony of all these phenomena of Nature. One of the most common motions of a body in the three-dimensional space is the combination of a rotation around an axis with a translation along that axis. The trajectory of every point (outside the axis) under this motion yields a helical curve. Let us present the angle of the rotation in such a motion in the form $\frac{p}{q}360^0$, where p, q are relatively small integers. It has been observed that for the spiral describing the distribution of leaves around a central stem in plants the fractions $\frac{p}{q}$ are formed by the consequent Fibbonacci numbers:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots,$$

which converge to the irrational number $(\sqrt{5}-1)/2$, which is the famous "golden mean".

It is worth noting that the above expression for the rotation angle in the spiral motion is similar to the resonance frequency in a two-mode system (e.g. in celestial mechanics), where p and q represent the order and the type of the resonance, respectively. This similarity can be interpreted as some kind of "resonance" in the world of plants.

Another example coming from physics is the geometry of convective motions. These motions give rise to the convective cells which could assume the shape of one-dimensional rollers or form a quadratic or hexagonal lattice (Benar cells). Let us consider the convection in a heated layer of liquid — the so-called Rayleigh-Benar convection.

Following Berge [14] we consider the horizontal layer of liquid (silicon oil) bounded from above and from below by rigid heat-conductive plates. If the Rayleigh number Ra (proportionate to the difference of temperatures in the vertical direction) exceeds certain critical value Ra_*, then sets in a convective motion taking the shape of rotating rollers. Notice, that the velocity field everywhere has zero projection onto the axis of the rollers (see Figs. 1.2, 1.3(a)). When Ra increases even further the structure persists until Ra exceeds another fixed value $Ra_{**} > Ra_*$. The third component of the velocity field becomes non-zero, and the secondary set of rollers appears, with axes perpendicular to those of the primary rollers (Fig. 1.3(b)). In Fig.1.3(b) one cane notice a similarity with a two-dimensional crystal. Values Ra_*, Ra_{**} are called bifurcational values.

Further increase of the Rayleigh number leads to the appearance of local contraction near the boundary (pinch effect) when $Ra = 20Ra_*$. If we now stop increasing Ra

[1] This shell is held by the dancing Shiva of the Hindu myth as one of the instruments through which he initiates creation (translator's note.)

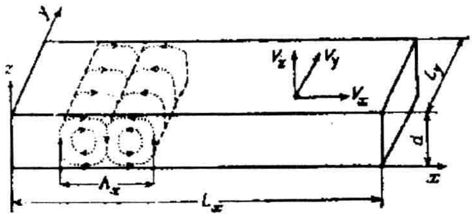

Fig. 1.2. Scheme of the convective motions of liquid in a rectangular configuration.

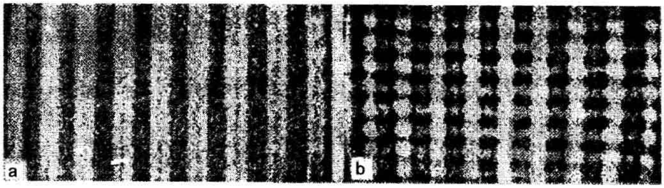

Fig. 1.3. Two-dimensional (a) and three-dimensional (b) structures.

and watch how the system evolves in time when $Ra \approx 20Ra_*$ if fixed (Fig. 1.4). Soon after the contraction begins there appears a isolated defect, around which the regular structure quickly disintegrates. This disintegration is followed by a metamorphosis which involves creation of a polygonal, cellular convective structure (Fig. 1.4 (d)). After 15 hours the entire structure disintegrates (melts) which is seen in Fig. 1.4 (f). Thereafter, the picture is changing continuously in a random way and the turbulence appears.

If we replaced the rectangular reservoir by a cylindrical one with diameter $D = 20d$, where d is the height of the cylinder, we would have got new invariant structures, some of which are shown in Fig. 1.5.

The so-called vortex structures are well known in hydrodynamics. They appear when a liquid flows round certain bodies. Analogous structures are known in aerodynamics. Let us at look at some experiments illustrating the vortex structures.

The following experiment was designed by Gak [16] (our discussion of its results below follows [20]). The experiment is based on the use of a magneto-hydrodynamic drive which allows to create a spatially periodic electromagnetic field in a thin layer of a weekly conducting liquid (electrolyte). Initially, the experiments of this type were conducted in a cuvette of dimensions 24 cm × 12 cm, 2–3 mm deep and with spatial period of 4.4 cm. By changing the density of the electric current between the

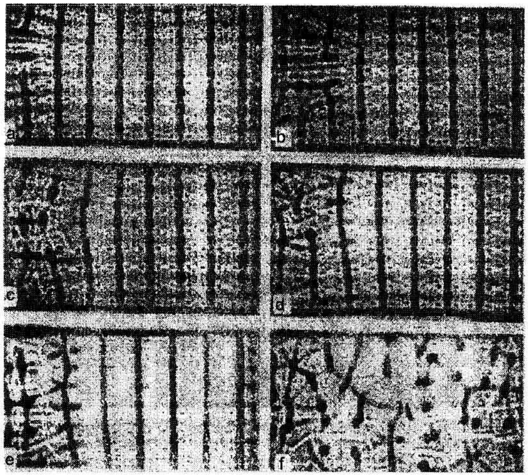

Fig. 1.4. Stages of "melting" of a three-dimensional structure at $Ra \approx 20Ra_8$.

Fig. 1.5. Convective structures in a cylindrical reservoir.

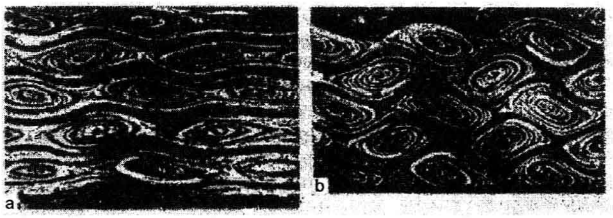

Fig. 1.6. Vortex structures for $Re/Re_{cr} = 1.1$(a) and $Re/Re_{cr} = 1.25$ (b).

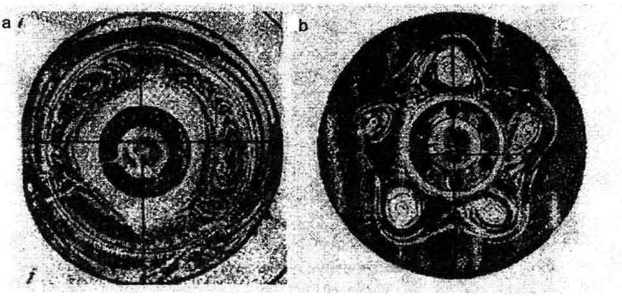

Fig. 1.7. Vortex structures in a round cuvette.

two electrodes immersed in the layer of liquid, one can create the flow with various Reinolds numbers (i.e. with different velocities of the liquid's movement). For some critical value of the Reinolds number (Re_{cr}) there appear "secondary" currents. In Fig. 1.6 the reader can see the picture of the secondary currents for $Re/Re_{cr} = 1.1$ (a) and $Re/Re_{cr} = 1.25$ (b).

Obukhov et al who experimented with a round cuvette also observed the secondary vortex currents [27, 20] (see Fig. 1.7).

The picture of the secondary currents presented in Fig. 1.7 (b) looks similar to the chain of vortices observed over Antarctica. The vortex structures shown in Figs. 1.6, 1.7 resemble the resonance structures in systems with 3/2 degrees of freedom (see Chapter 4).

Now let us talk about crystals. The most popular example of a crystal is a snowflake which has a hexagonal shape. In general, the symmetry in crystals can