fundamentals of Trigonometry

second edition

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Preface

This second edition differs from the first in that some of the chapters have been modified so that they better meet the needs and backgrounds of students who are deficient in trigonometry. In addition, most of the exercise sets have been changed.

Chapter One contains prerequisites for trigonometry—the real number system, functions, and graphs. As an aid to reviewing this material, more numerical problems than appeared in the first edition have been included. The concept of relation is defined, but is used primarily to introduce graphs of equations.

Chapters Two, Three and Four have the same format as in the previous edition; however, explanations of concepts which cause the most difficulty have been expanded and improved. Trigonometric functions of real numbers are introduced by means of the unit circle, and the more classical descriptions in terms of ratios are brought in shortly afterward. By blending these two approaches, instead of concentrating on one of them, the student should acquire a deeper understanding of this important class of functions.

The standard methods for solving oblique triangles are contained in Chapter Five. In Chapter Six, the ordered pair definition of complex numbers has been replaced by a definition involving the symbol $a + bi$. Many instructors favor this procedure and most students find it easier to understand. Of course, the only essential difference in the two definitions is the notation, provided one does not, at the outset, interpret the plus sign in $a + bi$ as addition.

Each section contains an ample supply of graded exercises. Exercise sets are designed so that by working either the odd-numbered or the even-numbered exercises, practice on all parts of the theory is obtained. There is a review section at the end of each chapter consisting of a list of important topics together with pertinent exercises. Answers to the odd-numbered exercises appear at the end of the text. An answer booklet is available for the even-numbered exercises.

This edition has benefited from comments and suggestions of users of the first edition. To these, and to all the people who have assisted in the formation of this text, I wish to express my sincere appreciation.

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chapter one Basic Concepts

In this chapter we discuss some important concepts which are prerequisites for the study of trigonometry. These include the terminology of sets, the real number system, graphs, and functions.

1 Sets

Throughout mathematics and other areas the concept of *set* is used extensively. This notion is so basic that we consider the word "set" a primitive term and do not attempt to define it formally. Intuitively, we think of a set as a collection of objects of some type. Thus one might speak of the set of books in a library, the set of giraffes in a zoo, the set of natural numbers 1, 2, 3, $\cdots$, the set of points in a plane, and so on. The objects in a given set are called the *elements* of the set. We assume that every set is *well defined* in the sense that there is some rule or property that can be used to determine whether a given object is or is not an element of the set.

Notationally, capital letters, A, B, C, R, S, $\cdots$ will be used to denote sets, whereas lower-case letters a, b, x, y, $\cdots$ will represent elements of sets. If S is a set, then the symbol $a \in S$ denotes the fact that a is an element of S. Similarly, $a, b \in S$ means that a and b are elements of S. The notation $a \notin S$ signifies that a is not an element of S.

If every element of a set S is also an element of a set T, then S is called a *subset* of T and we write $S \subseteq T$, or $T \supseteq S$, which may be read "S is contained in T" or "T contains S." For example, if T is the set of letters in the English alphabet and if S is the set of vowels, then $S \subseteq T$. It is important to note that for every set S we have $S \subseteq S$, since every element of S is an element of S. The symbol $S \nsubseteq T$ means that S is not a subset of T. In this case there is at least one element of S which is not an element of T.

We say that two sets S and T are *equal*, and write $S = T$, provided S and T contain precisely the same elements. This is equivalent to saying that $S \subseteq T$ and also $T \subseteq S$. If S and T are not equal, then we write $S \neq T$. If $S \subseteq T$ and $S \neq T$, then S is called a *proper* subset of T. In this case there exists at least one element of T which is not an element of S.

The notation $a = b$, translated "a equals b," means that a and b are symbols which represent the same element. For example, in arithmetic the symbol $2 + 3$ represents the same number as the symbol $4 + 1$ and hence we write $2 + 3 = 4 + 1$. Similarly, $a = b = c$ means that a, b, and c all represent the same element. Of course, $a \neq b$ means that a and b represent different elements. We assume that equality of elements of a set S satisfies the following three properties: (i) $a = a$ for all $a \in S$, (ii) if $a = b$, then $b = a$, and (iii) if $a = b$ and $b = c$, then $a = c$.

There are various ways of describing sets. One method, especially adapted for sets containing only a few elements, is to list all the elements within braces. For example, if S consists of the first five letters of the alphabet, we write $S = \{a, b, c, d, e\}$. When sets are given in this way, the order used in listing the elements is considered irrelevant. We could also write $S = \{a, c, b, e, d\}$, or $S = \{d, c, b, e, a\}$, and so on. This notation is also useful for describing larger sets when there is some definite pattern for the elements. As an illustration, we might specify the set $\mathbf{N}$ of *natural numbers* by writing

$$\mathbf{N} = \{1, 2, 3, 4, \cdots\},$$

where the dots may be read "and so on."

There is another convenient method for describing sets. If S consists of a set of elements from some set T, each of which has a certain property, denoted by p, then we write

$$S = \{x \in T \mid x \text{ has property } p\}$$

or, if the set T from which the elements are chosen is clear, then we merely write

$$S = \{x \mid x \text{ has property } p\}.$$

To translate this notation, for the braces read "the set of" and for the vertical bar read "such that." As a specific example, let $S = \{x \in \mathbf{N} \mid x + 2 = 7\}$. This can be read "$S$ is the set of elements x in $\mathbf{N}$ such that $x + 2 = 7$." Hence S contains only one element, the number 5. As another illustration, if $E = \{x \in \mathbf{N} \mid x \text{ is even}\}$, then E consists of the collection of all even natural numbers, that is, $E = \{2, 4, 6, 8, \cdots\}$. Another way of describing E is to write $E = \{2n \mid n \in \mathbf{N}\}$. The set F

of *odd* natural numbers can be denoted by $F = \{2n - 1 \mid n \in \mathbf{N}\}$. For example, by substituting the natural numbers 1, 2, 3, 4 for n, we obtain the elements 1, 3, 5, 7 of F.

One must distinguish between an element a of a set and the *set* consisting of the element a. Thus if $S = \{a, b, c\}$, then $a \in S$ *but not* $\{a\} \in S$. On the other hand, if we wish to discuss the *subset* of S consisting of the element a, we use the notation $\{a\}$ and write $\{a\} \subseteq S$.

The *empty set* $\varnothing$ is sometimes defined by $\varnothing = \{x \mid x \neq x\}$. The set $\varnothing$ differs from all other sets because it contains no elements. It is mainly a notational device we find convenient to use in certain instances. For example, if $S = \{x \in \mathbf{N} \mid x + 2 = 1\}$, then $S = \varnothing$, since $x + 2$ is never 1 when $x \in \mathbf{N}$. It is customary to assume that $\varnothing$ is a subset of every set S.

Let us list the subsets of the set $S = \{a, b, c\}$. There are 8 subsets in all. They are $\{a\}$, $\{b\}$, $\{c\}$, $\{a, c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, b, c\}$, and $\varnothing$. In Exercise 4 the student is asked to list the 16 subsets of a set which contains 4 elements.

When working with several sets, we assume that they are subsets of some larger set U, called a universal set. However, U will not always be given explicitly. If elements $x, y, z, \cdots$ are employed, without specifying any particular set, we assume they belong to some universal set U. With these remarks in mind, we state the following two definitions.

(1.1) Definition of Union

The *union* of two sets A and B, denoted by $A \cup B$, is the set

$$\{x \mid x \in A \text{ or } x \in B\}.$$

The word "or" in this definition and generally throughout mathematics means that either $x \in A$ or $x \in B$, or possibly that x is in *both* A and B.

(1.2) Definition of Intersection

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set

$$\{x \mid x \in A \text{ and } x \in B\}.$$

Thus the intersection of two sets consists of the elements which are *common* to the sets. If $A \cap B = \varnothing$, that is, if A and B have no elements in common, then A and B are said to be *disjoint*.

Example. If $A = \{a, b, c, d\}$, $B = \{b, c, e, f\}$ and $C = \{a, d\}$, find $A \cup B$, $A \cap B$, $A \cup C$, $A \cap C$, and $B \cap C$.

Solution: By (1.1) and (1.2) we have $A \cup B = \{a, b, c, d, e, f\}$, $A \cap B = \{b, c\}$, $A \cup C = A$, $A \cap C = C$, and $B \cap C = \varnothing$.

Sometimes sets are pictured by drawing circles, squares, or other simple closed curves in a plane, where it is understood that the points within these figures represent the elements of the sets. Thus if A and B are subsets of a set U, we might indicate this as in Fig. 1.1. Unions and

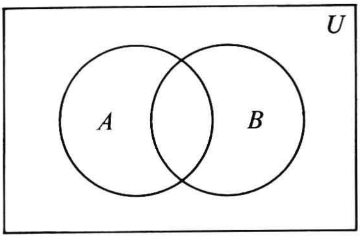

Figure 1.1

intersections can then be represented by shading appropriate parts of the figure. This is illustrated in Fig. 1.2, where we have deleted from

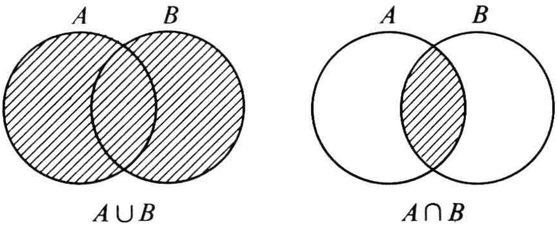

Figure 1.2

our picture the universal set U. The reader should also sketch $A \cup B$ and $A \cap B$ when $A \subseteq B$, $B \subseteq A$, or $A \cap B = \varnothing$. It is important to realize that diagrams such as these are used merely to help motivate and visualize notions concerning sets and are not used to prove or serve as steps in proofs of any theorems.

Unions and intersections of more than two sets can also be considered. For example, if A, B, and C are sets, we may consider the union of $A \cup B$ and C. Thus $(A \cup B) \cup C = \{x \mid x \in A \cup B \text{ or } x \in C\}$, that is, $(A \cup B) \cup C$ is the set of all elements x which appear in at least one of the sets A, B, or C. On the other hand, first forming $B \cup C$, we could consider $A \cup (B \cup C)$. Evidently $(A \cup B) \cup C = A \cup (B \cup C)$. Similarly, one can show that $(A \cap B) \cap C = A \cap (B \cap C)$.

Exercises

1 Use the notation of sets to designate each of the following:

(a) The set consisting of the last six letters of the alphabet.

(b) The set consisting of the letters of the alphabet which occur after the letter z.

(c) The natural numbers which are divisible by 10.

2 Use words to describe each of the following sets, where **N** is the set of natural numbers:

(a) $S = \{x \in \mathbf{N} \mid x \text{ is odd}\}$.
(b) $T = \{x \in \mathbf{N} \mid x = 5\}$.
(c) $V = \{10n \mid n \in \mathbf{N}\}$.

3 If $A = \{1, 2, 3\}$ determine whether each of the following is true or false and give reasons for your answers:

(a) $3 \in A$. (b) $3 \subseteq A$. (c) $\{3\} \subseteq A$.
(d) $A \subseteq A$. (e) $\varnothing \in A$. (f) $\varnothing \subseteq A$.

4 Find 16 different subsets of the set $W = \{1, 2, 3, 4\}$.

5 Find $A \cup B$ and $A \cap B$ if A and B are as follows:

(a) $A = \{2, 5, 3\}$, $B = \{3, 1, 6\}$.
(b) $A = \{a, b, c, d\}$, $B = \{d, e, a\}$.
(c) $A = \{a, b, c\}$, $B = \{d, e, f\}$.
(d) $A = \{1, 2\}$, $B = \{1, 2, 3\}$.

6 If $R = \{1, 5, 6\}$, $S = \{2, 3, 4\}$, and $T = \{6, 2\}$ find:

(a) $R \cap (S \cup T)$. (b) $T \cap (R \cup S)$.
(c) $R \cup (S \cap T)$. (d) $(R \cup S) \cap (R \cup T)$.

7 If A and B are sets, find $A \cap B$ if:

(a) $A \subseteq B$. (b) $B \subseteq A$.
(c) A and B are disjoint. (d) $A = B$.

8 Same as Exercise 7 for $A \cup B$.

9 Use overlapping circles in a plane to represent sets R, S, and T. Shade the region which represents each of the following:

(a) $R \cup (S \cup T)$. (b) $(R \cap S) \cap T$. (c) $R \cup (S \cap T)$.
(d) $R \cap (S \cup T)$. (e) $(R \cap S) \cup (R \cap T)$.

10 Prove that if A and B are sets, then $A \cup B = B$ if and only if $A \subseteq B$.

2 Real Numbers

The set used most frequently in mathematics is the set **R** of real numbers. We refer to **R**, together with the various properties possessed by its elements, as the *real number system*. The reader is undoubtedly well acquainted with symbols such as 2, $-\frac{3}{5}$, $\sqrt{3}$, 0, -8.614, $.3333 \cdots$, etc. which are used to denote elements of **R**. In this section we shall list some properties of **R** and review the notation and terminology associated with real numbers.

The system **R** is *closed* relative to operations of addition (denoted by "+") and multiplication (denoted by "·"). Thus, for every $a, b \in \mathbf{R}$, there corresponds a unique element $a + b$ called the *sum* of a and b and a unique element $a \cdot b$ (sometimes written ab) called the *product* of a and

b. These operations have the following properties, where all lower case letters denote arbitrary elements of **R**.

(1.3) Commutative Laws

$$a + b = b + a, \quad ab = ba.$$

(1.4) Associative Laws

$$a + (b + c) = (a + b) + c, \quad a(bc) = (ab)c.$$

(1.5) Distributive Laws

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc.$$

(1.6) Identity Elements

There exist special real numbers, denoted by 0 and 1, with the following properties:

$$a + 0 = a = 0 + a, \quad a \cdot 1 = a = 1 \cdot a.$$

(1.7) Inverse Elements

For every real number a, there is a real number denoted by $-a$ such that $a + (-a) = 0 = (-a) + a$. For every real number $a \neq 0$, there is a real number denoted by $1/a$ such that $a(1/a) = 1 = (1/a)a$.

A set which satisfies the above properties is referred to as a *field*. For this reason, (1.3)–(1.7) are sometimes called the *field properties* of the real number system.

The special real numbers 0 and 1 are referred to as *zero* and *one*, respectively. We call $-a$ the *negative* (or *additive inverse*) of a and $1/a$ the *reciprocal* (or *multiplicative inverse*) of a. The symbol a^{-1} is often used in place of $1/a$.

Many properties of **R** can be derived from (1.3)–(1.7). For example, one can prove the *cancellation laws*, which state that if $a + c = b + c$, then $a = b$, and if $ac = bc$, where $c \neq 0$, then $a = b$. One can also show that $a \cdot 0 = 0 = 0 \cdot a$ for every $a \in \mathbf{R}$. Moreover, if $ab = 0$, then either $a = 0$ or $b = 0$. The following rules involving negatives may be established: $-(-a) = a$, $(-a)b = -ab = a(-b)$, $(-a)(-b) = ab$ and $(-1)a = -a$.

If $a, b \in \mathbf{R}$, then the operation of *subtraction* (denoted by "$-$") is defined by $a - b = a + (-b)$. If $b \neq 0$, then *division* (denoted by "$\div$") is defined by $a \div b = a(1/b) = ab^{-1}$. The symbol a/b is often used in place of $a \div b$, and we refer to it as the *quotient of a by b* or the *fraction a over b*. The numbers a and b are called the *numerator* and *denominator*, respectively, of the fraction. It is important to note that a/b is not defined if $b = 0$, that is, *division by zero is not permissible in* **R**. The following rules for quotients may be established, where all denominators are nonzero real numbers:

$$\begin{aligned} a/b &= c/d \text{ if and only if } ad = bc, \\ (ad)/(bd) &= a/b, \\ (a/b) + (c/d) &= (ad + bc)/(bd), \\ (a/b)(c/d) &= (ac)/(bd). \end{aligned}$$

If we begin with the real number 1 and successively add it to itself, we obtain the set **N** of *positive integers* (also called the *natural numbers*):

$$\{1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, \cdots\}.$$

As usual, this set is specified by writing

$$\mathbf{N} = \{1, 2, 3, 4, \cdots\}.$$

The negatives $-1, -2, -3, -4, \cdots$ of the positive integers are referred to as *negative integers*. The set **Z** of *integers* is the totality of positive and negative integers together with the real number 0, that is,

$$\mathbf{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

A real number is called a *rational number* if it can be written in the form a/b, where a and b are integers and $b \neq 0$. Real numbers that are not rational are called *irrational*. The ratio of the circumference of a circle to its diameter is irrational. This real number, which is denoted by π is often *approximated* by the decimal 3.1416 or by the rational number $\frac{22}{7}$. We use the notation $\pi \doteq 3.1416$ to indicate that π is *approximately equal* to 3.1416.

Real numbers may be represented by *infinite decimals*. In order to obtain such a representation for rational numbers, the process of long division may be used. Thus, by long division, the decimal representation for the rational number $\frac{7434}{2310}$ is found to be $3.2181818 \cdots$, where the three dots indicate that the digits 1 and 8 repeat indefinitely. Such an expression is referred to as an *infinite repeating decimal*. Every rational number has associated with it an infinite repeating decimal and, conversely, given such a decimal it is possible to find a rational number which has, as its decimal representation, the given infinite repeating decimal. Decimal representations for irrational numbers may also be obtained; however, they are always *infinite nonrepeating*. For numerical