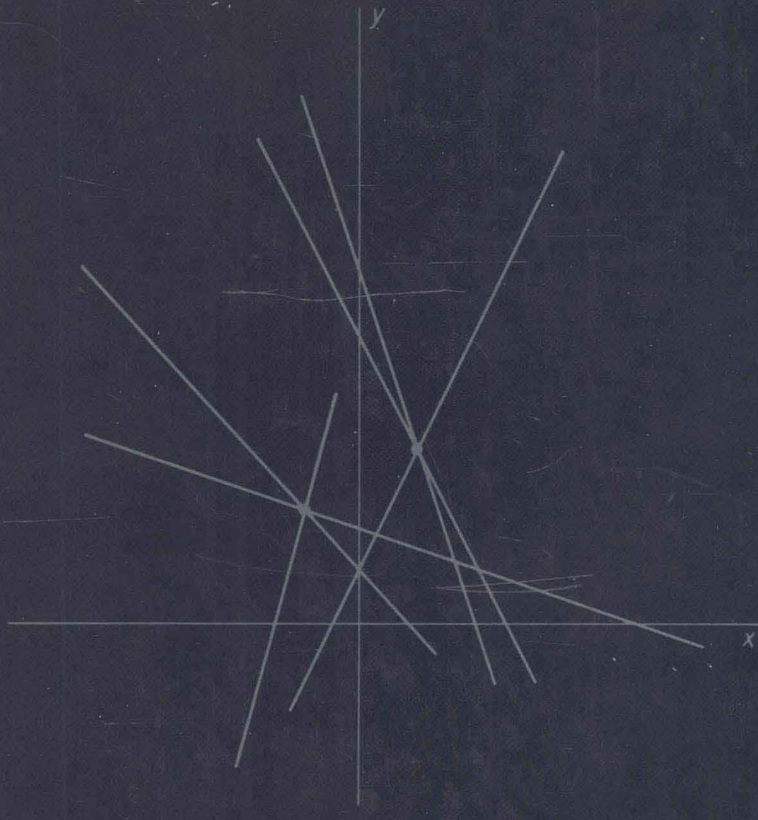
LINEAR ALGEBRA WITH APPLICATIONS



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BROOKS/COLE PUBLISHING COMPANY
Monterey, California
A Division of Wadsworth Publishing Company, Inc.

Consulting Editor: Robert J. Wisner, New Mexico State University

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging in Publication Data

Agnew, Jeanne
Linear algebra with applications.

Bibliography: p. 454
Includes index.
1. Algebras, Linear. I. Knapp, Robert C., joint author. II. Title.
QA184.A36 512'.5 77-17373
ISBN 0-8185-0256-8

Acquisition Editor: *Craig F. Barth*
Manuscript Editor: *Phyllis Niklas*
Production Editor: *Stephen E. White*
Cover Design: *Jamie S. Brooks*
Illustrations: *Eldon J. Hardy*

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PREFACE

This book is written for and to the student who is taking a first course in linear algebra. Such a course is taught, ordinarily at the sophomore level, in two-year colleges, colleges, and universities. It is required not only of mathematics majors, but also of majors in computer science, certain branches of engineering, and the physical sciences. Frequently, it is elected by students in business, economics, or the social sciences. The intent of this book is to present the traditional material of elementary linear algebra in a way that will meet the needs of this diverse audience.

With this purpose in mind, a section of each chapter is devoted to presenting simple applications of the material discussed. There is, of course, no claim that all possible applications are included. The most important applications often require too much preparatory discussion for inclusion in a book of this type. Applications that are included illustrate how the material can be put to use in many varied fields. They are designed to help satisfy the natural desire to ask "Is what I'm studying good for something?" They also encourage the student to "think applications" while proceeding in the text—a point of view that helps anyone understand and appreciate the meaning of mathematical ideas.

Along with the applications, some computer programs written in BASIC are given so that more involved computations can be carried out. These programs make it possible to work more realistic problems. More important, working through them helps the student understand the principles behind the computations.

Because linear algebra is the first course in which a student thinks of sets of data as objects themselves, this book begins with a chapter devoted to matrices. Definitions are motivated by examples that manipulate sets of data in reasonable ways. The student is encouraged to think of n-tuples as entities. The interpretation of powers of a matrix in various ways leads to a type of application that is stimulating to a beginning student.

The second chapter is devoted to systems of linear equations. The student is encouraged to think of the solution as a set of n-tuples and to write solutions as linear combinations of n-tuples.

In the third chapter, square matrices are discussed. A discussion of determinants and inverses of square matrices is followed by a section on the eigenvalues and eigenvectors of a matrix.

The introduction of eigenvalues at this stage is, in our opinion, one of the main strengths of the way we have arranged the topics in this book. It prevents the frustration felt by both students and instructors when, after a semester of working toward this topic, there is too little time left to do it justice. More important, the introduction of eigenvalues and eigenvectors at this point gives meaning to and motivation for the more theoretical topics in the latter part of the book.

In Chapter 4, linear spaces are introduced, and R^n is covered in detail. In this context, the concepts of linear independence, basis, dimension, and the null space of a matrix are introduced. At each stage, these topics are related to the study of eigenvalues and eigenvectors.

Chapter 5 is a treatment of linear transformations and the use of matrices in representing such transformations.

Chapter 6 is optional, but reinforces the student's understanding of Chapters 4 and 5 by using the ideas from these chapters in the context of polynomial spaces.

Chapter 7 returns to eigenvalues and diagonalization—this time from the point of view of linear transformations.

Chapter 8 gives a brief account of the special properties of symmetric matrices and the theory of real quadratic forms.

An instructor who wishes to emphasize the more theoretical aspects of the subject can omit the applications and computer programs without disturbing the continuity of the text. Even if the instructor should elect to do this, the inclusion of these topics in this book gives the interested student a chance to read them individually. If the instructor wishes to emphasize the computational aspects, there is ample material to build a complete course out of the first three chapters and parts of Chapters 4, 5, and 7.

The style of this book is informal. Examples are plentiful. Important results are labeled *Statement*. Usually, an argument to support the statement is included, but no attempt is made to dwell unduly on the formality of presenting a concise and complete proof. It is hoped that this approach will help the student concentrate on understanding *why* the statement is true, rather than on a formal way of writing. As an additional aid to organization and review, a short summary is included at the end of each section.

A special feature of the presentation that can be very helpful to the student is the inclusion throughout the text of Terribly Easy Matrix Problems, called *TEMPs*. These give an immediate check of the student's understanding of definitions and ideas. They supplement the examples and have greater student appeal, since they challenge the reader immediately to try his or her own skill before checking the answer.

In addition, each section is provided with extensive exercise sets. Exercises are of value only to those who work them. We cannot emphasize the importance of the exercises enough—the student should attempt as many of them as possible. Mastery of linear algebra can come only by conscientiously developing problem solving (and problem posing) skills.

In general, this book is written to the student. As such, it could be used for independent study or for correspondence study. However, in the opinion of the authors, no text, however well written, can do for the student what can be done by an inspired teacher. We hope that the point of view taken in this presentation will add to the enjoyment of both student and teacher as they work together in the fascinating study of linear algebra and its applications.

The authors wish to thank Patrick J. Bibby, Miami-Dade Community College, South Campus; Leroy J. Dickey, University of Waterloo; David Rodabaugh, University of Missouri; and Maurice D. Weir, Naval Postgraduate School, Monterey, for their many helpful suggestions. We are especially indebted to Bernard W. Levinger, Colorado State University, who read the manuscript with care and patience. His thoughtful and perceptive analysis of it and appreciation of the goals of our presentation made his comments especially valuable.

Also, we wish to thank Robert J. Wisner, Series Editor, for his encouragement and help during the preparation of the manuscript, the fine staff of Brooks/Cole for their efficient work in assembling it, and Herkimer County Community College for the generous provision of computer facilities.

Finally, we express our great appreciation to Judy Magnuson Knapp. In addition to performing her role as combined critic and cheering section, she worked all the exercises, prepared the answer section, and assisted in the correction of proofs. The authors, however, claim entire credit for any errors that may remain.

Jeanne Agnew
Robert C. Knapp

CONTENTS

CHAPTER 1

THE MATRIX AND HOW TO OPERATE WITH IT

1.1 THE MATRIX AND ADDITION

Everyone has seen a matrix. A matrix is simply a rectangular array of numbers such as might be found in a table. Throughout this book, the numbers that appear in matrices will be real numbers, and we will assume the ordinary rules of the arithmetic of real numbers.

EXAMPLE 1.1

The price of a swim suit depends on its style, size range, material, and pattern. The following table lists the price information in dollars for Speedo swim suits:

Type of swim suit	Male, 22–28	Male, 30–38	Female, 24–28	Female, 30–38
Regular (solid)	\$ 5.00	\$ 5.25	\$ 8.50	\$10.00
Regular (stripe)	5.75	6.25	10.00	12.00
Regular (print)	6.50	9.00	12.00	13.50
Lycra (solid)	8.50	9.25	14.50	16.50
Lycra (print)	10.85	11.50	18.00	25.50

The data are arranged in five rows. Each row refers to a particular type of suit and lists prices in different size ranges for that style. Once the sizes are arranged in order, this order is not changed.

The rectangular array of numbers in this table is the matrix

$$\begin{bmatrix} 5.00 & 5.25 & 8.50 & 10.00 \\ 5.75 & 6.25 & 10.00 & 12.00 \\ 6.50 & 9.00 & 12.00 & 13.50 \\ 8.50 & 9.25 & 14.50 & 16.50 \\ 10.85 & 11.50 & 18.00 & 25.50 \end{bmatrix}.$$

Although the headings on the table are valuable in interpreting the data, for purposes of matrix theory we are interested only in the array of numbers and in the operations between such arrays considered as entities.

The matrix in this example has five rows and four columns. The rows of the matrix, corresponding to the rows of the table, are ordered sets of four numbers, or ordered 4-tuples. Each column of the table lists prices in a fixed size range. The corresponding column of the matrix is an ordered 5-tuple.

Our purpose in this chapter is to establish definitions for adding and multiplying matrices and to study the laws that govern addition and multiplication. To accomplish this, we must understand certain terms. The vocabulary illustrated in Example 1.1 is stated formally in Definition 1.1.

DEFINITION 1.1

A *matrix* is a rectangular array of real numbers called the *elements* of the matrix. The plural of matrix is *matrices*.

Each horizontal array of elements in a matrix is called a *row*.

Each vertical array of elements in a matrix is called a *column*.

The number of rows and columns determines the *size* of a matrix. If there are m rows and n columns, the matrix is of *size m by n*, written $m \times n$.

If $m = n$, the matrix is said to be *square*.

A single real number is called a *scalar*.

An ordered set of n real numbers is called an *n-tuple*, written $(a_1, a_2, \ldots, a_n)$. If $n = 2$, the set is called a *pair* (in preference to *2-tuple*), and if $n = 3$, the set is called a *triple* (in preference to *3-tuple*).

Because we want to think of an entire set of numbers at one time, single letters are used to designate these sets. Single lowercase letters are used to designate n-tuples. When the real numbers in an ordered set are written out, they are enclosed in parentheses and separated by commas. To distinguish the n-tuple of real numbers from the individual real numbers, boldface type is used in this book for the letters that represent n-tuples. [Since you cannot write in boldface, you may want to add a bar above the letters to correspond to boldface type. Thus, the printed statement $\mathbf{u} = (1, 3, 5)$ would be written $\bar{u} = (1, 3, 5)$.]

Single capital letters are used to designate matrices, and again boldface type is used. In this way, *arrays* of numbers are distinguished from numbers themselves. The elements of a matrix are real numbers. If we want to designate a particular element, we describe its location by using subscripts. The first subscript tells the row in which it occurs, and the second subscript tells the column. The element in the ith row and jth column of a matrix is written a_{ij}. If the elements of a matrix are written out, the array is enclosed in square brackets.

An n-tuple of real numbers can be written as a $1 \times n$ array, or row. If $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ is written as a row, it is designated $\mathbf{U}_{1 \times n}$ and written $\mathbf{U}_{1 \times n} = [a_1 \ a_2 \ \cdots \ a_n]$. An n-tuple of real numbers can also be written as an $n \times 1$ array, or column. If the n-tuple $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ is written as a column, it is called $\mathbf{U}_{n \times 1}$ and written

$$\mathbf{U}_{n \times 1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

If the context makes clear whether a row or column is required, the subscript $1 \times n$ or $n \times 1$ is omitted.

Notation. The matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is of size $m \times n$. To shorten the statement, we write $\mathbf{A} = [a_{ij}]$ or $\mathbf{A} = [a_{ij}]_{m \times n}$ when we want to emphasize the size of $\mathbf{A}$.

The rows of an $m \times n$ matrix are n-tuples written as horizontal arrays. The first row is the n-tuple $\mathbf{u}_1 = (a_{11}, a_{12}, \ldots, a_{1n})$ written as the horizontal array $\mathbf{U}_1 = [a_{11} \ a_{12} \ \cdots \ a_{1n}]$. The ith row is the n-tuple $\mathbf{u}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ written as the horizontal array $\mathbf{U}_i = [a_{i1} \ a_{i2} \ \cdots \ a_{in}]$. The columns of an $m \times n$ matrix are m-tuples written as vertical arrays. The jth column is the m-tuple

$$\mathbf{v}_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$$

written as the $m \times 1$ array

$$\mathbf{V}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Throughout this book we will pause from time to time to ask some questions that will test your understanding of new vocabulary or concepts. *Do not read further until you have answered them.* Check your answers with the answers given. Since these problems are supposed to impede your progress only temporarily, we call them TEMPs (*T*erribly *E*asy *M*atrix *P*roblems).

TEMP 1.1

$$\text{Let } \mathbf{A} = \begin{bmatrix} \frac{1}{2} & 2 & 4 \\ 0 & 6 & -3 \end{bmatrix}.$$

1. Write the third column of **A**.
2. **A** has size ______ $\times$ ______.
3. What is a_{23}? What is a_{32}?
4. If $\mathbf{u} = (1, 5)$, what is $\mathbf{U}_{1\times 2}$? What is $\mathbf{U}_{2\times 1}$?

---

1. $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$; 2. 2×3; 3. $a_{23} = -3$, there is no a_{32}; 4. $\mathbf{U}_{1\times 2} = [1 \;\; 5]$, $\mathbf{U}_{2\times 1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

It is easier to remember definitions if they make sense. Think about a matrix as a collection of information. When would we want to declare that two matrices are equal? Only if they are identical; that is, when the information contained in them is exactly the same and presented in the same order.

DEFINITION 1.2

The matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are *equal* if and only if they are the same size and $a_{ij} = b_{ij}$ for each i and j.

In words, Definition 1.2 says that two matrices are equal only when they have the same number of rows, the same number of columns, and corresponding entries are the same.

TEMP 1.2

1. Let $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ 1 & 2 \end{bmatrix}$. What are the values of b_{11} and b_{12} if $\mathbf{A} = \mathbf{B}$?
2. Let $\mathbf{C} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 2 & c_{23} \end{bmatrix}$. Is there a choice of c_{23} for which $\mathbf{C} = \mathbf{A}$?

1. $b_{11} = -1$, $b_{12} = 0$; 2. **C** and **A** are not the same size, so they cannot be equal

EXAMPLE 1.2

The information contained in the table in Example 1.1 can also be presented in the following form:

	Regular (solid)	Regular (stripe)	Regular (print)	Lycra (solid)	Lycra (print)
Male, 22–28	$ 5.00	$ 5.75	$ 6.50	$ 8.50	$10.85
Male, 30–38	5.25	6.25	9.00	9.25	11.50
Female, 24–28	8.50	10.00	12.00	14.50	18.00
Female, 30–38	10.00	12.00	13.50	16.50	25.50

This form leads to the matrix

$$\mathbf{B} = \begin{bmatrix} 5.00 & 5.75 & 6.50 & 8.50 & 10.85 \\ 5.25 & 6.25 & 9.00 & 9.25 & 11.50 \\ 8.50 & 10.00 & 12.00 & 14.50 & 18.00 \\ 10.00 & 12.00 & 13.50 & 16.50 & 25.50 \end{bmatrix}.$$

The matrix **B** contains the same information as the matrix in Example 1.1, but the rows of **B** are the columns of the earlier matrix. We certainly cannot declare that the two arrays are equal according to our definition, but we emphasize the relationship between them by calling one array the *transpose* of the other.

DEFINITION 1.3

If **A** is an $m \times n$ array, the *transpose* of **A**, written $\mathbf{A}^{\mathrm{T}}$, is the $n \times m$ array formed as follows: The n rows of $\mathbf{A}^{\mathrm{T}}$ are the n columns of **A** and the m columns of $\mathbf{A}^{\mathrm{T}}$ are the m rows of **A**. Thus, $\mathbf{A}^{\mathrm{T}}$ is formed from **A** by interchanging rows with columns.

EXAMPLE 1.3

If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 5 \end{bmatrix}$.

If $\mathbf{U} = [1 \quad 2 \quad 3]$, then $\mathbf{U}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Also, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}^T = [1 \quad 4 \quad 3]$.

TEMP 1.3

1. Write $\mathbf{A}^T$ if $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$.
2. What can you say about $(\mathbf{A}^T)^T$ for any matrix $\mathbf{A}$?

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1. $\mathbf{A}^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{bmatrix}$; 2. $(\mathbf{A}^T)^T = \mathbf{A}$

EXAMPLE 1.4

In Example 1.1, the 4-tuple (5.00, 5.25, 8.50, 10.00) represents the cost of regular solid-color swim suits in different sizes. The cost of Lycra solid-color suits is represented by (8.50, 9.25, 14.50, 16.50). If each member of a school's swim team is furnished a solid-color regular suit for practice and a solid-color Lycra suit for competition, what would be the cost per child in each size range? For a boy sized 22–28, it would be $5.00 + 8.50 = \$13.50$. For a boy sized 30–38, the cost would be $5.25 + 9.25 = \$14.50$. The cost of both suits in each size range could be represented as a 4-tuple obtained by adding the original 4-tuples *elementwise*: (13.50, 14.50, 23.00, 26.50).

Suppose a family has a boy sized 32 and a girl sized 26 on the swim team. What will be the cost to the family of one suit for each child? The prices for the boys' suits are found in the 5-tuple that is the second column of the matrix in Example 1.1: (5.25, 6.25, 9.00, 9.25, 11.50). The prices for the girl's suits are found in the 5-tuple (8.50, 10.00, 12.00, 14.50, 18.00). The cost of one suit each, of comparable type, is the 5-tuple obtained by adding corresponding elements: (13.75, 16.25, 21.00, 23.75, 29.50).

Addition of n-tuples makes sense when they contain comparable information in the same order. In this case, it is reasonable to define the *sum* of two n-tuples to be the n-tuple formed by *adding elementwise*.

DEFINITION 1.4

Let $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{v} = (b_1, b_2, \ldots, b_n)$. The *sum* of **u** and **v** is the n-tuple $\mathbf{u} + \mathbf{v} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.

EXAMPLE 1.5

Let $\mathbf{v} = (5.00, 5.75, 6.50, 8.50, 10.85)$. Here, **v** is the 5-tuple that forms the first column of Example 1.1 and represents the cost of different types of boys' suits sized 22–28. What would we mean by $10\mathbf{v}$? If the team has ten boys in the size range 22–28, the array $10\mathbf{v}$ should give the cost of suits of each type for all ten of the boys:

$$10\mathbf{v} = (50.00, 57.50, 65.00, 85.00, 108.50).$$

The 5-tuple $10\mathbf{v}$ is obtained from the 5-tuple **v** by multiplying each element of **v** by 10.

DEFINITION 1.5

Let $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and let k be a scalar. Then the product of the n-tuple **u** and the scalar k is the n-tuple $k\mathbf{u} = (ka_1, ka_2, \ldots, ka_n)$.

In words, this definition says that the result of multiplying an n-tuple by a scalar k (a real number) is the n-tuple in which each element is k times the corresponding element in the original n-tuple.

TEMP 1.4

Let $\mathbf{u} = (1, 1, -3)$ and $\mathbf{v} = (0, 1, -1)$. Calculate $\mathbf{u} + \mathbf{v}$, $2\mathbf{u}$, $3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

$(1, 2, -4)$; $(2, 2, -6)$; $(0, 3, -3)$; $(2, 5, -9)$

Suppose **A** and **B** are matrices. When and how should $\mathbf{A} + \mathbf{B}$ be defined, and what should be the meaning of $k\mathbf{A}$ for a scalar k and a matrix **A**? Example 1.6 suggests definitions for addition of matrices and for multiplication of a matrix by a scalar.

EXAMPLE 1.6

Two sections of Matrix Theory are taught at Space University. One has 31 students and the other has 25. The students are sophomores and juniors