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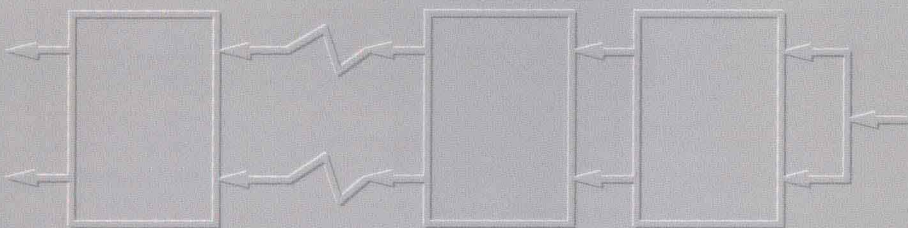
Orthogonal Rational Functions



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This book generalizes the classical theory of orthogonal polynomials on the complex unit circle or on the real line to orthogonal rational functions whose poles are among a prescribed set of complex numbers.

The first part treats the case where these poles are all outside the unit disk or in the lower half plane. Classical topics such as recurrence relations, numerical quadrature, interpolation properties, Favard theorems, convergence, asymptotics, and moment problems are generalized and treated in detail. The same topics are discussed for the different situation where the poles are located on the unit circle or on the extended real line. In the last chapter, several applications are mentioned including linear prediction, Pisarenko modeling, lossless inverse scattering, and network synthesis.

This theory has many applications in both theoretical real and complex analysis, approximation theory, numerical analysis, system theory, and in electrical engineering.

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CAMBRIDGE MONOGRAPHS ON APPLIED AND COMPUTATIONAL MATHEMATICS

5 Orthogonal Rational Functions

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Contents

List of symbols

$\mathbb{C}$	the complex plane $\mathbb{C} = \{z = \operatorname{Re} z + \mathbf{i} \operatorname{Im} z\}$
$\overline{\mathbb{C}}$	the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
$\mathbb{R}, \overline{\mathbb{R}}$	the real line $\mathbb{R} = \{z \in \mathbb{C} : \operatorname{Im} z = 0\}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ 16
$\mathbb{Z}$	the integers
$\mathbb{U}, \mathbb{L}, \mathbb{H}$	$\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathbb{L} = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, $2\,\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$
$\mathbb{D}, \mathbb{T}, \mathbb{E}$	$\mathbb{D} = \{z \in \mathbb{C} : \lvert z\rvert < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} : \lvert z\rvert = 1\}$, $\mathbb{E} = \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$\mathbb{O}, \partial\mathbb{O}, \mathbb{O}^e$	$\mathbb{O}$ region in $\mathbb{C}$: $\mathbb{D}$ or $\mathbb{U}$ $\partial\mathbb{O}$ boundary of $\mathbb{O}$: $\mathbb{T}$ or $\mathbb{R}$ $\mathbb{O}^e$ exterior of $\mathbb{O}$: $\mathbb{E}$ or $\mathbb{L}$ 16
$\mathring{\mu}$	normalized measure on $\overline{\partial\mathbb{O}}$: $d\mathring{\mu}(t) = d\mu(t)/(1 + t^2)$ on $\overline{\mathbb{R}}$, $d\mathring{\mu}(t) = d\mu(t)$ on $\mathbb{T}$ 20
$d\lambda, d\mathring{\lambda}$	normalized Lebesgue measures: $d\lambda(t) = (\pi)^{-1} dt$ for $\mathbb{R}$, $d\lambda(t) = (2\pi)^{-1} d\theta$, $t = e^{\mathbf{i}\theta}$ for $\mathbb{T}$ .... 17 $d\mathring{\lambda}(t) = d\lambda(t)/(1 + t^2)$ on $\mathbb{R}$, $d\mathring{\lambda}(t) = d\lambda(t)$ on $\mathbb{T}$ 19
$H(X)$	functions holomorphic in X 16
$H_p, N, \mathring{H}_p$	Hardy and Nevanlinna classes................. 17, 20 $\mathring{H}_p = H_p(d\mathring{\lambda})$ for $\mathbb{R}$, $\mathring{H}_p = H_p(d\lambda)$ for $\mathbb{T}$ 20
$\langle f, g\rangle_{\mathring{\mu}}$	inner products: $\int_{-\pi}^{\pi} f(e^{\mathbf{i}\theta})\overline{g(e^{\mathbf{i}\theta})}\, d\mu(\theta)$ for $\mathbb{D}$, $\int_{\overline{\mathbb{R}}} f(t)\overline{g(t)}\, d\mathring{\mu}(t)$ for $\mathbb{U}$ 17

$\hat{z}$	reflection in the boundary:	
	$\hat{z} = 1/\overline{z}$ for $\mathbb{D}$, $\hat{z} = \overline{z}$ for $\mathbb{U}$	20
$f_*(z)$	substar conjugate: $f_*(z) = \overline{f(\hat{z})}$	20
$\alpha_k, k \geq 0$	basic interpolation points:	
	$\alpha_k \in \partial\mathbb{O}$ for boundary case	257
	$\alpha_k \in \mathbb{O}$ otherwise.	43
$A_n, \hat{A}_n$	$A_n = \{\alpha_1, \ldots, \alpha_n\}$, $\hat{A}_n = \{\hat{\alpha}_1, \ldots, \hat{\alpha}_n\}$	44
$A, \hat{A}$	$A = \{\alpha_1, \alpha_2, \ldots\}$, $\hat{A} = \{\hat{\alpha}_1, \hat{\alpha}_2, \ldots\}$	44
A_n^w, A_n^0	$A_n^w = \{w, \alpha_1, \ldots, \alpha_n\}$	133
	$A_n^0 = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$	44
$\mathbb{O}_0, \mathbb{O}_0^e$	$\mathbb{O}_0 = \mathbb{O} \setminus A_n$, $\mathbb{O}_0^e = \mathbb{O}^e \setminus A_{n*}$	241
α_0	special point: $\alpha_0 = 0$ for $\mathbb{D}$, $\alpha_0 = \mathbf{i}$ for $\mathbb{U}$	19
	in the boundary case: $\alpha_0 = -1$ for $\mathbb{T}$, $\alpha_0 = \infty$ for $\mathbb{R}$	257
$\varpi_w(z), \varpi_i(z)$	$\varpi_w(z) = 1 - \overline{w}z$ for $\mathbb{D}$, $\varpi_w(z) = z - \overline{w}$ for $\mathbb{U}$,	
	$\varpi_i = \varpi_{\alpha_i}$	19
$\pi_n(z)$	$\pi_n(z) = \prod_{k=1}^n \varpi_k(z)$	44
$D(t, z), E(t, z)$	$D(t, z)$ Riesz–Herglotz–Nevanlinna kernel:	
	$D(t, z) = (t + z)/(t - z)$ for $\mathbb{D}$	27
	$D(t, z) = -\mathbf{i}(1 + tz)/(t - z)$ for $\mathbb{U}$	27
	$E(t, z) = 1 + D(t, z)$	83
$\mathcal{C}, \mathcal{B}$	$\mathcal{C}$ positive real functions:	
	$\mathcal{C} = \{f \in H(\mathbb{O}) : f(\mathbb{O}) \subset \mathbb{H}\}$	23
	$\mathcal{B}$ bounded analytic functions:	
	$\mathcal{B} = \{f \in H(\mathbb{O}) : f(\mathbb{O}) \subset \mathbb{D}\}$	23
$\mathcal{A}$	$\mathcal{A} = \{[\Delta_1 \;\; \Delta_2] : \Delta_1, \Delta_2 \in H(\mathbb{O})$, $\Delta_2(z) \neq 0, z \in \mathbb{O}, \Delta_1/\Delta_2 \in \mathcal{B}\}$	141
$\Omega_\mu(z) \in \mathcal{C}$	Riesz–Herglotz–Nevanlinna transform:	
	$\Omega_\mu(z) = \mathbf{i}c + \int D(t, z)\, d\mathring{\mu}(t)$, $c \in \mathbb{R}$	27
$C(t, z)$	Cauchy kernel:	
	$C(t, z) = [\varpi_0(\alpha_0)\varpi_{z*}(t)]^{-1}$ for $\mathbb{O}$,	
	$C(t, z) = t/(t - z)$ for $\mathbb{D}$,	
	$C(t, z) = 1/[2\mathbf{i}(t - z)]$ for $\mathbb{U}$	23
$P(t, z)$	Poisson kernel:	
	$P(t, z) = [\varpi_z(z)/\varpi_0(\alpha_0)]/[\varpi_z(t)\varpi_{z*}(t)]$ for $\mathbb{O}$	27
	$P(t, z) = (1 - \lvert z\rvert^2)/\lvert t - z\rvert^2$ for $\mathbb{D}$ if $t \in \mathbb{T}$,	27
	$P(t, z) = \operatorname{Im} z/\lvert t - z\rvert^2$ for $\mathbb{U}$ if $t \in \mathbb{R}$	27
M_w	Möbius transform:	
	$M_w(z) = (z - w)/(1 - \overline{w}z)$	25

$\sigma(z)$	spectral factor: $\sigma(z) = c\ \exp\{\frac{1}{2}\int D(t,z)\log\mu'(t)\,d\mathring{\lambda}(t)\}$, $c \in \mathbb{T}$ 33
$\mathcal{P}_n, n \geq 0, \mathcal{P}_\infty, \mathcal{P}$	polynomials of degree $\leq n$
	$\mathcal{P}_{-n} = \mathcal{P}_{n*} = \{p : p_* \in \mathcal{P}_n\}$ 16
	$\mathcal{P}_\infty = \cup_{n=0}^\infty \mathcal{P}_n,\ \mathcal{P} = \text{closure}(\mathcal{P}_\infty)$ 149
$p^*, p \in \mathcal{P}_n$	superstar conjugate for polynomials: $p^*(z) = z^n p_*(z)$ for $\mathbb{D}$, $p^*(z) = p_*(z)$ for $\mathbb{U}$ 54
$\alpha_\emptyset$	"forbidden" point in boundary case: $\alpha_\emptyset = 1$ for $\mathbb{T}$, $\alpha_\emptyset = 0$ for $\mathbb{R}$ 257
β	special point boundary case: $\beta = 0$ for $\mathbb{T}$, $\beta = \mathbf{i}$ for $\mathbb{R}$ 257
z_i	convergence factors for Blaschke products: $z_i = -\overline{\alpha}_i/\lvert\alpha_i\rvert$ for $\mathbb{D}$, $z_i = \lvert 1+\alpha_i^2\rvert/(1+\alpha_i^2)$ for $\mathbb{U}$, $z_i = 1$ for $\alpha_i = \alpha_0$ 43
ζ_α, ζ_i	Blaschke factors: $\zeta_\alpha(z) = \varpi_\alpha(z)^*/\varpi_\alpha(z),\ \zeta_i(z) = z_i\zeta_{\alpha_i}(z)$ 43
B_n	finite Blaschke products: $B_0 = 1,\ B_n(z) = \prod_{i=1}^n \zeta_i(z), n \geq 1,$ 43
$Z_i,\ \ i \geq 0$	basic factors for boundary case: $Z_i = b(z)/[\varpi_i(z)/\varpi_i(\alpha_\emptyset)]$ on $\partial\mathbb{O}$, $Z_i = \mathbf{i}(1-z)(1-\alpha_i)/(z-\alpha_i)$ on $\mathbb{T}$, $Z_i = z/(1-z/\alpha_i)$ on $\mathbb{R}$ 259
$b(z)$	numerator of basic factors in boundary case: $b(z) = \mathbf{i}\varpi_\emptyset(z)/\varpi_\emptyset(\alpha_0)$ on $\partial\mathbb{O}$, $b(z) = \mathbf{i}(1-z)$ on $\mathbb{T}$, $b(z) = z$ on $\mathbb{R}$ 259
b_n	basis functions for boundary case: $b_0 = 1, b_n(z) = \prod_{i=1}^n Z_i(z), n \geq 1$
$\mathcal{L}_n, \mathcal{L}_\infty, \mathcal{L}$	fundamental spaces:
	$\mathcal{L}_n = \text{span}\{b_k : k = 0,\ldots,n\}$ for boundary case 257
	$\mathcal{L}_n = \text{span}\{B_k : k = 0,\ldots,n\}$ otherwise 43
	$\mathcal{L}_\infty = \cup_{n=0}^\infty \mathcal{L}_n,\ \mathcal{L} = \text{closure}(\mathcal{L}_\infty)$ 149
$\mathcal{L}_n(w)$	$\mathcal{L}_n(w) = \{f : f \in \mathcal{L}_n : f(w) = 0\}$ 60
$f^*, f \in \mathcal{L}_n$	superstar conjugate in $\mathcal{L}_n$: $f^*(z) = B_n(z)f_*(z)$ 53
$\phi_n, \varphi_n, \Phi_n$	ϕ_n orthonormal functions for $\mathcal{L}_n$: $\langle\phi_k,\phi_l\rangle_{\mathring{\mu}} = \delta_{kl}$
	φ_n monic orthogonal functions: $\varphi_n = \phi_n/\kappa_n$ 55
	Φ_n rotated orthogonal functions: $\Phi_n = \epsilon_n\phi_n$ 80

κ_n, κ_n'	κ_n leading coefficient of ϕ_n: $\kappa_n = \overline{\phi_n^*(\alpha_n)}$ 55
	boundary case: $\phi_n(z) = \phi_n(\alpha_\emptyset) + \cdots +$
	$\kappa_n' b_{n-1}(z) + \kappa_n b_n(z)$ 260
ψ_n, Ψ_n	ψ_n functions of the second kind for $\mathcal{L}_n$
	$\psi_n(z) = \int [E(t,z)\phi_n(t) - D(t,z)\phi_n(z)]\, d\mathring{\mu}(t)$ 83
	$\Psi_n = \epsilon_n \psi_n$ 100
	boundary case:
	$\psi_n(z) = \int D(t,z)[\phi_n(t) - \phi_n(z)]\, d\mathring{\mu}(t)$
	$- \delta_{n0} Z_0(z)/Z_0(\beta)$ 267
$k_n(z,w), K_n(z,w)$	$k_n(z,w)$ reproducing kernels for $\mathcal{L}_n$:
	$k_n(z,w) = \sum_{i=0}^n \phi_i(z)\overline{\phi_i(w)}$ 55
	$K_n(z,w)$ normalized reproducing kernels for $\mathcal{L}_n$:
	$K_n(z,w) = k_n(z,w)/k_n(w,w)^{1/2}$ 70
$P_n(z), Q_n(z)$	Q_n: para-orthogonal functions,
	$Q_n(z) = Q_n(z,\tau) = \phi_n(z) + \tau\phi_n^*(z)$, $\tau \in \mathbb{T}$ 109
	P_n: associated functions of second kind,
	$P_n(z) = P_n(z,\tau) = \psi_n(z) - \tau\psi_n^*(z)$ 111
	boundary case: quasi-orthogonal functions,
	$Q_n(z) = Q_n(z,\tau) = \phi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}^*(z)$,
	$\tau \in \overline{\mathbb{R}}$ 280
	P_n: associated functions of second kind,
	$P_n(z) = P_n(z,\tau) = \psi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \psi_{n-1}^*(z)$ 291
$s_w(z) = s(z,w)$	Szegő kernel:
	$s(z,w) = [\varpi_0(z)\overline{\varpi_0(w)}]/[\overline{\varpi_0(\alpha_0)\sigma(w)}\varpi_w(z)\sigma(z)]$
	for $\mathbb{O}$
	$s(z,w) = 1/[\overline{\sigma(w)}(1 - \overline{w}z)\sigma(z)]$ for $\mathbb{D}$
	$s(z,w) = [(z+\mathbf{i})(\mathbf{i} - \overline{w})]/[2\mathbf{i}\overline{\sigma(w)}(z - \overline{w})\sigma(z)]$
	for $\mathbb{U}$ 61
$S_w(z) = S(z,w)$	$S_w(z) = s_w(z)/\sqrt{s_w(w)}$ 177
$\mathcal{L}_{p,q},\ p,q \geq 0$	$\mathcal{L}_{p,q} = \mathcal{L}_q \cdot \mathcal{L}_{p*}$, $\mathcal{L}_{0,n} = \mathcal{L}_n$, $\mathcal{L}_{n,0} = \mathcal{L}_{-n} =$
	$\mathcal{L}_{n*}$ 106
$\mathcal{R}_n, \mathcal{R}_\infty, \mathcal{R}$	$\mathcal{R}_n = \mathcal{L}_{n,n}$ 112
	$\mathcal{R}_\infty = \cup_{n=0}^\infty \mathcal{R}_n$, $\mathcal{R} = \text{closure}(\mathcal{R}_\infty)$ 149
$\mu_n \xrightarrow[n]{*} \mu$	Convergence in weak star topology 226
$\lim_{z \mapsto \alpha} f(z)$	Orthogonal limit to $\alpha \in \overline{\partial\mathbb{O}}$
	$\lim_{r\uparrow 1} f(r\alpha)$ for $\mathbb{T}$
	$\lim_{y\downarrow 0} f(\alpha + \mathbf{i}y)$ for $\mathbb{R}$
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Introduction

This monograph forms an introduction to the theory of orthogonal rational functions. The simplest way to see what we mean by orthogonal rational functions is to consider them as generalizations of orthogonal polynomials.

There is not much confusion about the meaning of an orthogonal polynomial sequence. One says that $\{\phi_n\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence if ϕ_n is a polynomial of degree n and it is orthogonal to all polynomials of lower degree. Thus given some finite positive measure μ (with possibly complex support), one considers the Hilbert space $L_2(\mu)$ of square integrable functions that contains the polynomial subspaces $\mathcal{P}_n$, $n = 0, 1, \ldots$. Then $\{\phi_n\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence if $\phi_n \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ and $\phi_n \perp \mathcal{P}_{n-1}$. In particular, when the support of the measure is (part of) the real line or of the complex unit circle, one gets the most widely studied cases of such general orthogonal polynomials. Such orthogonal polynomials appear of course in many different aspects of theoretical analysis and applications. The topics that are central in our generalization to rational functions are moment problems, quadrature formulas, and classical problems of complex approximation in the complex plane.

Polynomials can be seen as rational functions whose poles are all fixed at infinity. For the orthogonal rational functions, we shall fix a sequence of poles $\{\gamma_k\}_{k=1}^{\infty}$, which, in principle, can be taken anywhere in the extended complex plane. Some of these γ_k can be repeated, possibly an infinite number of times, or they could be infinite. However, the sequence is fixed once and for all and the order in which the γ_k occur (possible repetitions included) is also given. This will then define the n-dimensional spaces of rational functions $\mathcal{L}_n$ that consist of all the rational functions of degree n whose poles are among $\gamma_1, \ldots, \gamma_n$ (including possible repetitions). We then consider $\{\phi_n\}_{n=0}^{\infty}$ to be a sequence of orthogonal rational functions if $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and $\phi_n \perp \mathcal{L}_{n-1}$.

There are two possible generalizations, depending on whether one generalizes the polynomials orthogonal on the real line or the polynomials orthogonal on the unit circle. The difference lies in the location of the finite poles that are introduced in the rational case. In the case of the circle, the pole at infinity is outside the closed unit disk. There, it is the most natural choice to introduce finite poles that are all outside the closed unit disk. This guarantees that the rational functions are analytic at least inside the unit disk, which allows us to transfer many properties from the polynomial to the rational case. Moreover, if the poles are not on the circle, then we avoid difficulties that could arise from singularities of the integrand in the support of the measure.

If the support of the measure is, however, contained in the real line, then the pole at infinity may be in the (closure of) the support of the measure. The most natural generalization is here to choose finite poles that are on the real line itself, that is, possibly in the support of the measure for which orthogonality is considered.

Of course one can by a Cayley transform map the unit circle to the (extended) real line and the open unit disk to the upper half plane. Since this transform maps rational functions to rational functions, it makes sense to consider the analog of the orthogonal rational functions on the unit circle with poles outside the closed disk, which are the orthogonal rational functions orthogonal on the real line with poles in the lower half plane. Conversely, one can consider the orthogonal rational functions with poles on the unit circle and that are orthogonal with respect to a measure supported on the unit circle as the analog of orthogonal rational functions on the real line with poles on the real line.

The cases of the real line and the unit circle, which are linked by such a Cayley transform, are essentially the same and can be easily treated in parallel, which we shall do in this monograph. The distinction between the case where the poles are outside or inside the support of the measure is, however, substantial. We have chosen to give a detailed and extensive treatment in several chapters of the case where the poles are outside the support. The case where the poles are in the support (which we call the boundary case) is treated more compactly in a separate chapter.

This brief sketch should have made clear in what sense these orthogonal rational functions generalize orthogonal polynomials. Now, what are the results of the polynomial case that have been generalized to the rational case? As we suggested above, we do not go into the details of all kinds of special orthogonal polynomials by imposing a specific measure or weight function. We do keep generality by considering arbitrary measures, but we restrict ourselves to measures supported on the real line or the unit circle. In that sense we are not as general as the "general orthogonal polynomials" in the book of Stahl and Totik [193].

Orthogonal polynomials have now been studied so intensely that many different and many detailed results are available. It would be impossible to give in one volume the generalizations of all these to the rational case. We have opted for an introduction to the topic and we give only generalizations of classical interpolation problems of Schur and Carathéodory type, of quadrature formulas, and of moment problems. There is a certain logic in this because interpolation problems are intimately related to quadrature formulas and these quadrature formulas are an essential tool for solving the moment problems.

These connections were made clear and were used explicitly in the book by Akhiezer [2], which treats "the classical moment problem." To some extent we have followed a similar path for the rational case.

First, we derive a recurrence relation for the orthogonal rational functions. In our setup, this is mainly based on a Christoffel–Darboux type relation. In the boundary case, this recurrence generalizes the three-term recurrence relation of orthogonal polynomials; in the case where the poles are outside the support of the measure, this is a generalization of the Szegő recurrence relation.

To describe all the solutions of the recurrence relation, a second, independent solution is considered, which is given by the sequence of associated functions of the second kind.

These functions of the second kind appear as numerators and the orthogonal rational functions as denominators in the approximants of a continued fraction that is associated with the recurrence relation. The continued fraction converges to the Riesz–Herglotz–Nevanlinna transform of the measure and the approximants interpolate this function in Hermite sense. This is the interpolation problem that we alluded to. It is directly related to the algorithm of Nevanlinna–Pick, which is a (rational) multipoint generalization of the Schur algorithm that relates to the polynomial case.

A combination of the orthogonal rational functions and the associated functions of the second kind give another solution of the recurrence relation called the quasi-orthogonal or para-orthogonal functions in the boundary or nonboundary case respectively. It can be arranged that these functions have simple zeros that are on the real line or on the unit circle. These zeros are used as the nodes of quadrature formulas. In the nonboundary case, such n-point quadrature formulas are optimal in the sense that corresponding weights can be chosen in such a way that the quadrature formulas have the largest possible domain of validity. For the boundary case, these quadrature formulas are "nearly optimal" in general. Their domain of validity has a dimension one less than the optimal one. However, when the zeros of the orthogonal rational functions themselves happen to be a good choice, then the quadrature formula is really optimal. In the polynomial case, this corresponds to Gaussian quadrature formulas on the real line or Szegő quadrature formulas for the circle.

These quadrature formulas are an essential tool in the construction of a solution of the moment problems. These moment problems are rational generalizations of the polynomial case that correspond to the Hamburger moment problem in the case of the real line and the trigonometric moment problem in the case of the circle.

Two other aspects are important or are at least closely connected to the solution of these moment problems. First, there is the well-known fact that, as n goes to ∞, the polynomial spaces $\mathcal{P}_n$ become dense in the Hardy spaces H_p. A similar result will only hold for the spaces $\mathcal{L}_n$ under certain conditions for the poles. Second, there is the general question of asymptotics for the orthogonal rational functions, for the interpolants, for the quadrature formulas, etc., when n tends to infinity. Such results were extensively studied in the polynomial case. We shall devote a large chapter to their generalizations.

After this general introduction, let us have a look at the roots of this theory, at the applications in which it was used, and let us have a closer look at the technical difficulties that arise by lifting the polynomial to the rational case. Since the central theme up to Chapter 10 is the generalization of results related to Szegő polynomials, orthogonal on the unit circle, let us take these as a starting point.

The particularly rich and fascinating theory of polynomials orthogonal on the unit circle needs no advertising. These polynomials are named after Szegő since his pioneering work on them. His book on orthogonal polynomials [196] was first published in 1939, but the ideas were already published in several papers in the 1920s. The Szegő polynomials were studied by several authors. For example, they play an important role in books by Geronimus [94], Freud [87], Grenander and Szegő [102], and several more recent books on orthogonal polynomials.

It is also in Szegő's book that the notion of a reproducing kernel is clearly introduced. Later on these became a studied object of their own. The book by Meschkowski [148] is a classic. In our exposition, reproducing kernels take a rather important place and the Christoffel–Darboux summation formula, which expresses the nth reproducing kernel in terms of the nth or $(n+1)$st orthogonal polynomials (in our case rational functions), is used again and again in many places throughout our monograph.

Szegő's interest in polynomials orthogonal on the unit circle was inspired by the investigation of the eigenvalue distribution of Toeplitz forms, an even older subject related to coefficient problems as initiated by Carathéodory [48, 49] and Carathéodory and Fejér [50] and further discussed by F. Riesz [184, 185],