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L-Functions and Galois Representations

Edited by
David Burns, Kevin Buzzard and Jan Nekovář

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L-functions and Galois Representations

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DAVID BURNS,

KEVIN BUZZARD

AND

JAN NEKOVÁŘ

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Preface

The London Mathematical Society symposium on L-functions and Galois representations took place at the University of Durham from the 19th to the 30th of July, 2004; this book is a collection of research articles in the areas covered by the conference, in many cases written by the speakers or audience members. There were series of lectures in each of the following subject areas:

- Local Langlands programme
- Local p-adic Galois representations
- Modularity of Galois representations
- Automorphic forms and Selmer groups
- p-adic modular forms and eigenvarieties
- The André-Oort conjecture

In practice it is becoming harder to distinguish some of these areas from others, because of major recent progress, much of which is documented in this volume. As well as these courses, there were 19 individual lectures. The organisers would like to thank the lecturers, and especially those whom we persuaded to contribute to this volume.

The symposium received generous financial support from both the EPSRC and the London Mathematical Society. These symposia now command a certain reputation in the number theory community and the organisers found it easy to attract many leading researchers to Durham; this would not have been possible without the financial support given to us, and we would like to heartily thank both organisations.

The conference could not possibly have taken place if it had not been for the efforts of John Bolton, James Blowey and Rachel Duke of the Department of Mathematics at the University of Durham, and for the hospitality of Grey College. We are grateful to both these institutions for their help in making the operation run so smoothly.

The feedback from the participants to the organisers seemed to indicate that many participants found the symposium mathematically stimulating; and the organisers can only hope that this volume serves a similar purpose.

David Burns
Kevin Buzzard
Jan Nekovář

List of participants

Viktor Abrashkin (Durham)
Amod Agashe (Missouri)
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Richard Taylor (Harvard)
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Douglas Ulmer (Arizona)
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Otmar Venjakob (Heidelberg)
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Christian Wuthrich (Cambridge)
Andrei Yafaev (University College London)
Atsushi Yamagami (Kyoto)
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Stark–Heegner points and special values of L-series

Massimo Bertolini
Dipartimento di Matematica
Universita' degli Studi di Milano
Via Saldini 50 20133
Milano, Italy
Massimo.Bertolini@mat.unimi.it

Henri Darmon
McGill University Mathematics Department,
805 Sherbrooke Street West Montreal,
QC H3A-2K6 CANADA
darmon@math.mcgill.ca

Samit Dasgupta
Department of Mathematics,
Harvard University,
1 Oxford St,
Cambridge, MA 02138, U.S.A.,
dasgupta@math.harvard.edu

Introduction

Let E be an elliptic curve over $\mathbb{Q}$ attached to a newform f of weight two on $\Gamma_0(N)$. Let K be a real quadratic field, and let $p\|N$ be a prime of multiplicative reduction for E which is inert in K, so that the p-adic completion K_p of K is the quadratic unramified extension of $\mathbb{Q}_p$.

Subject to the condition that all the primes dividing $M := N/p$ are split in K, the article [Dar] proposes an analytic construction of "Stark–Heegner points" in $E(K_p)$, and conjectures that these points are defined over specific class fields of K. More precisely, let

$$R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[1/p]) \text{ such that } M \text{ divides } c \right\}$$

be an Eichler $\mathbb{Z}[1/p]$-order of level M in $M_2(\mathbb{Q})$, and let $\Gamma := R_1^\times$ denote the group of elements in R of determinant 1. This group acts by Möbius transformations on the K_p-points of the p-adic upper half-plane

$$\mathcal{H}_p := \mathbb{P}^1(K_p) - \mathbb{P}^1(\mathbb{Q}_p),$$

and preserves the non-empty subset $\mathcal{H}_p \cap K$. In [Dar], modular symbols attached to f are used to define a map

$$\Phi : \Gamma \backslash (\mathcal{H}_p \cap K) \longrightarrow E(K_p), \tag{0.1}$$

whose image is conjectured to consist of points defined over ring class fields of K. Underlying this conjecture is a more precise one, analogous to the classical Shimura reciprocity law, which we now recall.

Given $\tau \in \mathcal{H}_p \cap K$, the collection $\mathcal{O}_\tau$ of matrices $g \in R$ satisfying

$$g \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda_g \begin{pmatrix} \tau \\ 1 \end{pmatrix} \quad \text{for some } \lambda_g \in K, \tag{0.2}$$

is isomorphic to a $\mathbb{Z}[1/p]$-order in K, via the map $g \mapsto \lambda_g$. This order is also equipped with the attendant ring homomorphism $\eta : \mathcal{O}_\tau \longrightarrow \mathbb{Z}/M\mathbb{Z}$ sending g to its upper left-hand entry (taken modulo M). The map η is sometimes referred to as the *orientation* at M attached to τ. Conversely, given any $\mathbb{Z}[1/p]$-order $\mathcal{O}$ of discriminant prime to M equipped with an orientation η, the set $\mathcal{H}_p^{\mathcal{O}}$ of $\tau \in \mathcal{H}_p$ with associated oriented order equal to $\mathcal{O}$ is preserved under the action of Γ, and the set of orbits $\Gamma \backslash \mathcal{H}_p^{\mathcal{O}}$ is equipped with a natural simply transitive action of the group $G = \mathrm{Pic}^+(\mathcal{O})$, where $\mathrm{Pic}^+(\mathcal{O})$ denotes the narrow Picard group of oriented projective $\mathcal{O}$-modules of rank one. Denote this action by $(\sigma, \tau) \mapsto \tau^\sigma$, for $\sigma \in G$ and $\tau \in \Gamma \backslash \mathcal{H}_p^{\mathcal{O}}$. Class field theory identifies G with the Galois group of the *narrow ring class field* of K attached to $\mathcal{O}$, denoted H_K. It is conjectured in [Dar] that the points $\Phi(\tau)$ belong to $E(H_K)$ for all $\tau \in \mathcal{H}_p^{\mathcal{O}}$, and that

$$\Phi(\tau)^\sigma = \Phi(\tau^\sigma), \qquad \text{for all } \sigma \in \mathrm{Gal}(H_K/K) = \mathrm{Pic}^+(\mathcal{O}). \tag{0.3}$$

In particular it is expected that the point

$$P_K := \Phi(\tau_1) + \cdots + \Phi(\tau_h)$$

should belong to $E(K)$, where $\tau_1, \ldots, \tau_h$ denote representatives for the distinct orbits in $\Gamma \backslash \mathcal{H}_p^{\mathcal{O}}$. The article [BD3] shows that the image of P_K in $E(K_p) \otimes \mathbb{Q}$ is of the form $t \cdot \mathbf{P}_K$, where

(i) t belongs to $\mathbb{Q}^\times$;
(ii) $\mathbf{P}_K \in E(K)$ is of infinite order precisely when $L'(E/K, 1) \neq 0$;

provided the following ostensibly extraneous assumptions are satisfied

(i) $\bar{P}_K = a_p P_K$, where $\bar{P}_K$ is the Galois conjugate of P_K over K_p, and a_p is the pth Fourier coefficient of f.
(ii) The elliptic curve E has at least two primes of multiplicative reduction.

The main result of [BD3] falls short of being definitive because of these two assumptions, and also because it only treats the image of P_K modulo the torsion subgroup of $E(K_p)$.

The main goal of this article is to examine certain "finer" invariants associated to P_K and to relate these to special values of L-series, guided by the analogy between the point P_K and classical Heegner points attached to imaginary quadratic fields.

In setting the stage for the main formula, let $E/\mathbb{Q}$ be an elliptic curve of conductor M; it is essential to assume that all the primes dividing M are *split* in K. This hypothesis is very similar to the one imposed in [GZ] when K is imaginary quadratic, where it implies that $L(E/K, 1)$ vanishes systematically because the sign in its functional equation is -1. In the case where K is real quadratic the "Gross-Zagier hypothesis" implies that the sign in the functional equation for $L(E/K, s)$ is 1 so that $L(E/K, s)$ vanishes to even order and is expected to be frequently non-zero at $s = 1$. Consistent with this expectation is the fact that the Stark–Heegner construction is now unavailable, in the absence of a prime $p\|M$ which is inert in K.

The main idea is to bring such a prime into the picture by "raising the level at p" to produce a newform g of level $N = Mp$ which is *congruent* to f. The congruence is modulo an appropriate ideal λ of the ring $\mathcal{O}_g$ generated by the Fourier coefficients of g. Let A_g denote the abelian variety quotient of $J_0(N)$ attached to g by the Eichler-Shimura construction. The main objective, which can now be stated more precisely, is to relate the *local behaviour at* p of the Stark–Heegner points in $A_g(K_p)$ to the algebraic part of the special value of $L(E/K, 1)$, taken modulo λ.

The first key ingredient in establishing such a relationship is an extension of the map Φ of (0.1) to arbitrary eigenforms of weight 2 on $\Gamma_0(Mp)$ such as g, and not just eigenforms with rational Fourier coefficients attached to elliptic curves, in a precise enough form so that phenomena related to congruences between modular forms can be analyzed. Let $\mathbb{T}$ be the full algebra of Hecke operators acting on the space of forms of weight two on $\Gamma_0(Mp)$. The theory presented in Section 1, based on the work of the third author [Das], produces a torus T over K_p equipped with a natural $\mathbb{T}$-action, whose character group (tensored with $\mathbb{C}$) is isomorphic as a $\mathbb{T} \otimes \mathbb{C}$-module to the space of weight 2 modular forms on $\Gamma_0(Mp)$ which are new at p. It also builds a Hecke-stable lattice $L \subset T(K_p)$, and a map Φ generalising (0.1)

$$\Phi : \Gamma\backslash(\mathcal{H}_p \cap K) \longrightarrow T(K_p)/L. \tag{0.4}$$

It is conjectured in Section 1 that the quotient T/L is isomorphic to the rigid analytic space associated to an abelian variety J defined over $\mathbb{Q}$. A strong partial result in this direction is proven in [Das], where it is shown that T/L is isogenous over K_p to the rigid analytic space associated to the p-new quotient $J_0(N)^{p\text{-new}}$ of the jacobian $J_0(N)$. In Section 1, it is further conjectured that the points $\Phi(\tau) \in J(K_p)$ satisfy the same algebraicity properties as were stated for the map Φ of (0.1).

Letting Φ_p denote the group of connected components in the Néron model of J over the maximal unramified extension of $\mathbb{Q}_p$, one has a natural Hecke-equivariant projection

$$\partial_p : J(\mathbb{C}_p) \longrightarrow \Phi_p. \tag{0.5}$$

The group Φ_p is described explicitly in Section 1, yielding a concrete description of the Hecke action on Φ_p and a description of the primes dividing the cardinality of Φ_p in terms of "primes of fusion" between forms on $\Gamma_0(M)$ and forms on $\Gamma_0(Mp)$ which are new at p.

This description also makes it possible to attach to E and K an explicit element

$$\mathcal{L}(E/K,1)_{(p)} \in \bar{\Phi}_p,$$

where $\bar{\Phi}_p$ is a suitable f-isotypic quotient of Φ_p. Thanks to a theorem of Popa [Po], this element is closely related to the special value $L(E/K,1)$, and, in particular, one has the equivalence

$$L(E/K,1) = 0 \quad \Longleftrightarrow \quad \mathcal{L}(E/K,1)_{(p)} = 0 \text{ for all } p.$$

Section 2 contains an exposition of Popa's formula.

Section 3 is devoted to a discussion of $\mathcal{L}(E/K,1)_{(p)}$; furthermore, by combining the results of Sections 1 and 2, it proves the main theorem of this article, an avatar of the Gross-Zagier formula which relates Stark–Heegner points to special values of L-series.

Main Theorem. *For all primes p which are inert in K,*

$$\partial_p(P_K) = \mathcal{L}(E/K,1)_{(p)}.$$

Potential arithmetic applications of this theorem (conditional on the validity of the deep conjectures of Section 1) are briefly discussed in Section 4.

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1 Stark–Heegner points on $J_0(Mp)^{p\text{-new}}$

Heegner points on an elliptic curve E defined over $\mathbb{Q}$ can be defined analytically by certain complex line integrals involving the modular form

$$f := \sum_{n=1}^{\infty} a_n(E) e^{2\pi i n z}$$

corresponding to E, and the Weierstrass parametrization of E. To be precise, let τ be any point of the complex upper half plane $\mathcal{H} := \{z \in \mathbb{C} | \Im z > 0\}$. The complex number

$$J_\tau := \int_\infty^\tau 2\pi i f(z) dz \in \mathbb{C}$$

gives rise to an element of $\mathbb{C}/\Lambda_E \cong E(\mathbb{C})$, where Λ_E is the Néron lattice of E, and hence to a complex point $P_\tau \in E(\mathbb{C})$. If τ also lies in an imaginary quadratic subfield K of $\mathbb{C}$, then P_τ is a *Heegner point* on E. The theory of complex multiplication shows that this analytically defined point is actually defined over an abelian extension of K, and it furthermore prescribes the action of the Galois group of K on this point.

The Stark–Heegner points of [Dar], defined on elliptic curves over $\mathbb{Q}$ with multiplicative reduction at p, are obtained by replacing complex integration on $\mathcal{H}$ with a double integral on the product of a p-adic and a complex upper half plane $\mathcal{H}_p \times \mathcal{H}$.

We now very briefly describe this construction. Let E be an elliptic curve over $\mathbb{Q}$ of conductor $N = Mp$, with $p \nmid M$. The differential $\omega := 2\pi i f(z) dz$ and its anti-holomorphic counterpart $\bar{\omega} = -2\pi i f(\bar{z}) d\bar{z}$ give rise to two elements in the DeRham cohomology of $X_0(N)(\mathbb{C})$:

$$\omega^{\pm} := \omega \pm \bar{\omega}.$$

To each of these differential forms is attached a *modular symbol*

$$m_E^{\pm}\{x \to y\} := (\Omega_E^{\pm})^{-1} \int_x^y \omega^{\pm}, \qquad \text{for } x, y \in \mathbb{P}^1(\mathbb{Q}).$$

Here $\Omega_E^{\pm}$ is an appropriate complex period chosen so that $m_E^{\pm}$ takes values in $\mathbb{Z}$ and in no proper subgroup of $\mathbb{Z}$.

The group Γ defined in the Introduction acts on $\mathbb{P}^1(\mathbb{Q}_p)$ by Möbius transformations. For each pair of cusps $x, y \in \mathbb{P}^1(\mathbb{Q})$ and choice of sign $\pm$, a $\mathbb{Z}$-valued additive measure $\mu^{\pm}\{x \to y\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$ can be defined by

$$\mu^{\pm}\{x \to y\}(\gamma \mathbb{Z}_p) = m_E^{\pm}\{\gamma^{-1} x \to \gamma^{-1} y\}, \tag{1.1}$$

where γ is an element of Γ. Since the stabilizer of $\mathbb{Z}_p$ in Γ is $\Gamma_0(N)$, equation (1.1) is independent of the choice of γ by the $\Gamma_0(N)$-invariance of $m_E^{\pm}$. The motivation for this definition, and a proof that it extends to an additive measure on $\mathbb{P}^1(\mathbb{Q}_p)$, comes from "spreading out" the modular symbol $m_E^{\pm}$ along the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ (see [Dar], [Das], and Section 1.2 below). For any $\tau_1, \tau_2 \in \mathcal{H}_p$ and $x, y \in \mathbb{P}^1(\mathbb{Q}_p)$, a multiplicative double integral on $\mathcal{H}_p \times \mathcal{H}$ is then defined by (multiplicatively) integrating the function $(t-\tau_1)/(t-\tau_2)$ over $\mathbb{P}^1(\mathbb{Q}_p)$ with respect to the measure $\mu^{\pm}\{x \to y\}$:

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2}\int_x^y \omega_{\pm} := \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu^{\pm}\{x \to y\}(t)$$
$$= \lim_{||\mathcal{U}|| \to 0} \prod_{U \in \mathcal{U}} \left(\frac{t_U-\tau_2}{t_U-\tau_1}\right)^{\mu^{\pm}\{x\to y\}(U)} \in \mathbb{C}_p^{\times}. \quad (1.2)$$

Here the limit is taken over uniformly finer disjoint covers $\mathcal{U}$ of $\mathbb{P}^1(\mathbb{Q}_p)$ by open compact subsets U, and t_U is an arbitrarily chosen point of U. Choosing special values for the limits of integration, in a manner motivated by the classical Heegner construction described above, one produces special elements in $\mathbb{C}_p^{\times}$. These elements are transferred to E using Tate's p-adic uniformization $\mathbb{C}_p^{\times}/q_E \cong E(\mathbb{C}_p)$ to define Stark–Heegner points.

In order to lift the Stark–Heegner points on E to the Jacobian $J_0(N)^{p\text{-new}}$, one can replace the modular symbols attached to E with the universal modular symbol for $\Gamma_0(N)$. In this section, we review this construction of Stark–Heegner points on $J_0(N)^{p\text{-new}}$, as described in fuller detail in [Das].

1.1 The universal modular symbol for $\Gamma_0(N)$

The first step is to generalize the measures $\mu^{\pm}\{x \to y\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$. As we will see, the new measure naturally takes values in the p-new quotient of the homology group $H_1(X_0(N), \mathbb{Z})$. Once this measure is defined, the construction of Stark–Heegner points on $J_0(N)^{p\text{-new}}$ can proceed as the construction of Stark–Heegner points on E given in [Dar]. The Stark–Heegner points on $J_0(N)^{p\text{-new}}$ will map to those on E under the modular parametrization $J_0(N)^{p\text{-new}} \to E$.

We begin by recalling the universal modular symbol for $\Gamma_0(N)$. Let $\mathcal{M} := \mathrm{Div}_0\, \mathbb{P}^1(\mathbb{Q})$ be the group of degree zero divisors on the set of cusps of the complex upper half plane, defined by the exact sequence

$$0 \to \mathcal{M} \to \mathrm{Div}\, \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Z} \to 0. \quad (1.3)$$

The group Γ acts on $\mathcal{M}$ via its action on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformations.

For any abelian group G, a *G-valued modular symbol* is a homomorphism $m : \mathcal{M} \longrightarrow G$; we write $m\{x \to y\}$ for $m([x]-[y])$. Let $\mathcal{M}(G)$ denote the