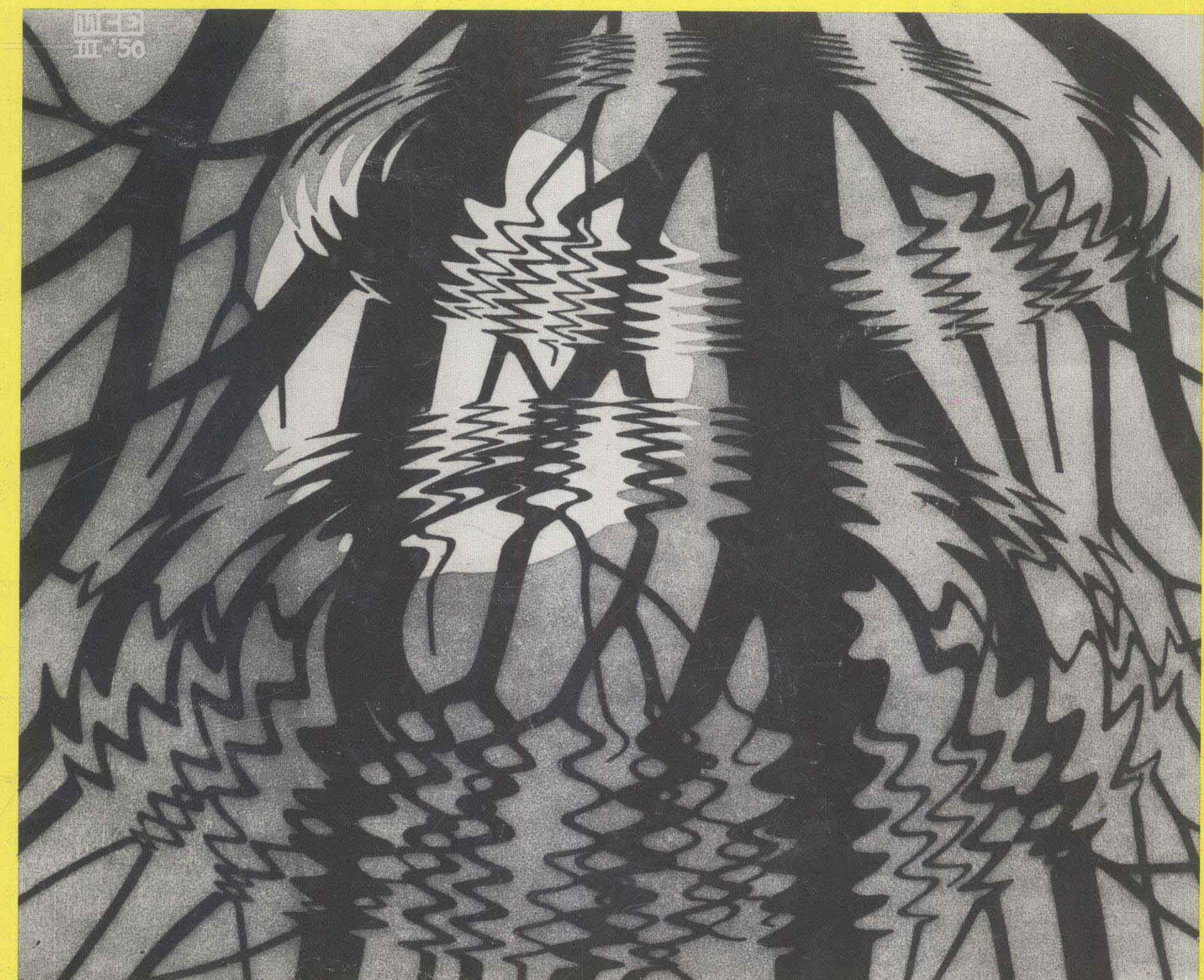
a survey of mathematics

Mario F. Triola



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a survey of mathematics

Mario F. Triola

Dutchess Community College
Poughkeepsie, New York

to my parents

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Cover: Rippled Surface *by the contemporary Dutch artist Maurits C. Escher (1898–1972). This print combines with other Escher works to reflect a strong interest in mathematics which the artist acquired as he experimented with visual perceptions. Just as Escher's* Rippled Surface *depicts a reflection, mathematics itself is a reflection of man's most precious ability—his power to reason. And as mathematics affected Escher's art, it also affects each of us and the world in which we live.*

preface

This book is intended for non-mathematics majors and non-science majors who require a broad knowledge of various branches of mathematics but lack extensive backgrounds. A knowledge of arithmetic is the only essential prerequisite. Some very basic principles of algebra are used in rare instances, but students having no algebra background can easily circumvent those occurrences. Also, Appendix A is a review of the fundamental principles of algebra.

The attempt has been made throughout the book to link mathematics with reality. The abundance of photographs and illustrations combine with the historical sections and exercises to provide a societal perspective which illuminates the practical value of mathematics. Chapters 1 and 2 on logic and set theory stress the importance of correct reasoning in making conclusions. Chapters 3 and 4 survey various structures of number systems and bases; a section on the metric system is also included. Chapters 5 and 6 survey the basic principles of probability and statistics while Chapter 7 deals with computers, flow charting, and programming. Chapter 8 begins with some basic ideas of conventional plane geometry and then briefly explores some key concepts of non-Euclidean geometries and topology.

Numerous examples and exercises are included in each chapter. Sections normally include non-routine "B" exercises that are more difficult or introduce new concepts.

The chapters are designed to be essentially independent so that topics can be covered in any order. Also, students encountering difficulty with one section will not be doomed for the remainder of the course. A typical selection for general liberal arts students might include Chapters 1–7. A course designed for elementary school teachers might stress Chapters 1–4 and Chapter 8, while a class of social science and business majors might stress Chapters 2, 5, 6, and 7.

I extend my sincere thanks to those who assisted with reviews and suggestions. In particular, I wish to thank the Cummings staff for their monumental help in the production of this book. Finally, I wish to thank my family who persevered and understood.

M.F.T.

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1

LOGIC

1 Introduction

The development of logic as a formal discipline can be broken down into three periods. The ancient Greeks were the first to really analyze the reasoning process. In the classical Greek period (approximately 600 B.C. to 300 B.C.), we find a class system consisting of an intellectual upper class who looked down upon the working class. Since, in their leisure, they were not concerned with practical applications of knowledge, the intellectual members of the upper class tended to emphasize theory and even to separate theory from practice. In this atmosphere of philosophy and theory, it was natural to insist upon deduction as the means of reasoning. Only deduction yields indubitable conclusions, and indubitable conclusions in the form of universal truths were what the ancient Greek intellectuals sought. The most outstanding member of this intellectual class was Aristotle (384–322 B.C.). Since he was a leader in developing logic as a separate branch of philosophy, he is called the father of traditional logic, which was based on the "syllogism." Here is a typical example of an Aristotelian syllogism.

> All men are mortal.
> Socrates is a man.
> Therefore, Socrates is mortal.

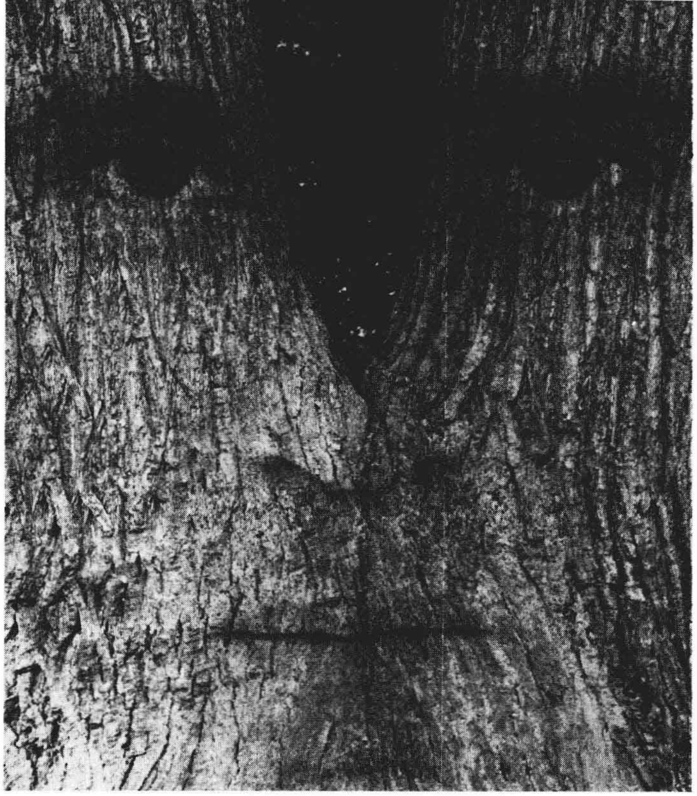
The system of logic developed in this period had some shortcomings. One major flaw was that no systematic and orderly scheme was developed to enable rapid deductions. The limited use of symbols hindered any further developments of logic.

The second major phase in the development of logic came at the time of Gottfried Wilhelm Leibniz who lived from 1646 to 1716. Leibniz is considered to be the first serious scholar of symbolic logic. He attempted to use symbolism in logic and ultimately to construct a universal language of reasoning. This new symbolic system of logic could then be applied to mathematical and scientific concepts. Because he was unable to break away from certain tenets of traditional or Aristotelian logic, Leibniz failed to provide a workable foundation for symbolic logic. However, he did renew interest in that direction.

The third and most significant phase in the development of logic came in the nineteenth century. The rapid growth of mathematics had caused an examination of the types of reasoning used to develop mathematical systems. Errors in logic made by earlier mathematicians were discovered, and the movement to correct past mistakes and to avoid new ones led mathematicians to investigate and further develop logic. A major new development was accomplished by George Boole.

George Boole was born in Lincoln, England, in 1815. Since his father was a shopkeeper, the existing social class system mandated that George Boole was destined to remain a member of the lower class. The level of education provided to Boole and his peers was not geared toward improving social or economic status. For example, some knowledge of Latin was considered to be a sign of distinction, but Latin was not taught in Boole's school. In striving to rise above his assigned station, George Boole studied and learned Latin and Greek on his own. At the age of twelve, he translated an ode of Horace into English, and his father entered it in the local paper. Because some serious errors became apparent, Boole was humiliated enough to spend the next two years independently studying Latin and Greek.

By the age of sixteen, Boole was required to provide for his parents. He spent the next four years teaching in elementary schools, and at the age of twenty he opened a school of his own. This was the turning point in the life of George Boole. He recognized that properly educated students must know some mathematics, yet he found the material in textbooks to be inadequate. Armed with only a knowledge of elementary mathematics and a lively interest, George Boole independently studied the works



A child returning home through darkening woods may incorrectly conclude that a tree is really a threatening ogre. It often happens that intuition and sense perceptions lead us to incorrect conclusions, but more reliable conclusions are attained through the proper use of logic.

of master mathematicians. He not only mastered these works but went on to make original contributions of his own!

Among Boole's friends was Augustus DeMorgan (1806–1871), who became involved in a controversy over logic with Sir William Hamilton (1788–1856), a Scottish philosopher. Boole, recognizing that DeMorgan was correct, tried to help by writing and publishing "The Mathematical Analysis of Logic," his first work on the subject. The brief but profound and perspicacious work won recognition for Boole. Finally, in 1849, George Boole was appointed Professor of Mathematics at Queen's College, Cork, Ireland. With more leisure time, and freedom from financial burdens, Boole was able to refine his work on logic. In 1854, he published *An Investigation of the Laws of Thought, On Which are Founded the Mathematical Theories of Logic and Probabilities.* In this masterpiece, Boole developed an "algebra" of logic; certain types of reasoning were reduced to the simple manipulations of symbols. The symbolism so necessary for the development of logic was finally supplied. Symbolic, or mathematical, logic was born with the publication of this significant work.

Boole married in 1855. In 1864, at the age of forty-nine, he died of pneumonia which he caught by keeping a lecture date after being soaked in a rainfall. He died after having received widespread recognition and knowing that he left behind a significant contribution: the development of symbolic logic. Logical reasoning could now be accomplished through the use of formulas and rules; manipulations were simplified by the substitution of concise symbols for cumbersome statements. Understanding became clearer. Relationships and patterns were more easily recognized. The development of logic could now accelerate.

2 Propositions and Logical Connectives

To begin our investigation into the fundamentals of symbolic logic, we should first consider the types of statements with which we will be dealing. Not every statement of the English language is applicable to symbolic logic.

Definition A *proposition* is a statement which is either true or false, but not both.

Symbolic logic deals only with propositions, that is, only those statements which must be either true or false. The following are examples of statements which are propositions.

1. The moon is made of green cheese.
2. Abraham Lincoln was the first president of the United States.
3. December 17, 1997, falls on a Sunday.
4. 3572111 is an even number.
5. All mathematicians are neurotic.

The following are examples of statements which are *not* propositions.

6. Please help me out.
7. Which way did you come in?
8. Stop and think.
9. $x = 4$.
10. Right on!

We cannot assign a value of "true" or "false" to any of statements 6 through 10. Statement 9 is false for some values of x and true for others; its truth or falsity cannot be determined as it stands. The principles of symbolic logic will not be applied to statements of this type. We will deal only with those declarative sentences which can be categorized, as they stand, as true or false.

In a system called "symbolic logic," a certain degree of symbolism is to be expected. Propositions will be denoted by letters such as p, q, r, s.

Examples

p: Jesse James was found to be guilty.
q: Jesse James spent thirty days in jail.
r: Jesse James paid a $30 fine.
s: Jesse James escaped.

In the development of any new mathematical system, we usually begin by learning about the types of *elements* upon which the system is based. For example, young children begin to learn about the system of arithmetic first by learning what numbers are; they learn to count. The second step usually involves learning the basic *operations* characteristic of the system. After knowing numbers and learning to count, the youngster then learns the operations of addition, subtraction, multiplication, and division. In learning the system of symbolic logic, we will follow a procedure similar to that of learning arithmetic. Just as numbers are the basic elements of arithmetic, propositions are the basic elements of symbolic logic. Knowing what propositions are, we will now move on to learn about the *operations* that we perform on the propositions.

In practical applications of logic, it has been found that most statements are complex and involve several propositions. Hence, it becomes necessary to devise a means of connecting propositions. This is accomplished through the use of *logical connectives.* The logical connectives are operations for symbolic logic just as addition and multiplication are operations for the system of arithmetic. (See Table 1.)

Once more, let p, q, r, s denote the propositions given in the above examples.

Table 1

Logical Connective (Operation)	English Equivalent	Symbol	Example of Use
conjunction	and	$\wedge$	$p \wedge q$
disjunction	or	$\vee$	$p \vee q$
negation	not; it is false that . . .	$\sim$	$\sim p$
conditional	if . . . , then . . .	$\rightarrow$	$p \rightarrow q$

Using the connectives, we can construct complex or compound propositions.

$p \wedge q$: Jesse James was found to be guilty and spent thirty days in jail.
$q \vee r$: Jesse James spent thirty days in jail or he paid a \$30 fine.
$p \rightarrow q$: If Jesse James was found guilty, then he spent thirty days in jail.
$p \rightarrow (q \vee r)$: If Jesse James was found to be guilty, then he spent thirty days in jail or paid a \$30 fine.
$\sim p \wedge \sim q$: Jesse James was not found to be guilty and did not spend thirty days in jail.

The following examples indicate some of the more difficult problems which are encountered in translating English statements into symbolic form, and vice versa.

Example It is not true that all radicals are violent.

p: A given person is a radical.
q: A given person is violent.

Solution The words "it is not true that" indicate that the remaining part of the proposition should be negated. Also, the part of the statement that "all radicals are violent" can be restated as "if a person is a radical, then he is violent." The given proposition can now be put into a more workable form: "It is not true that if a person is a radical, then he is violent."

$$\sim(p \to q)$$

Just as $p \to q$ symbolically represents "if p then q," the statements below are also represented by $p \to q$.

p only if q
p implies q
p is a sufficient condition for q
q is a necessary condition for p

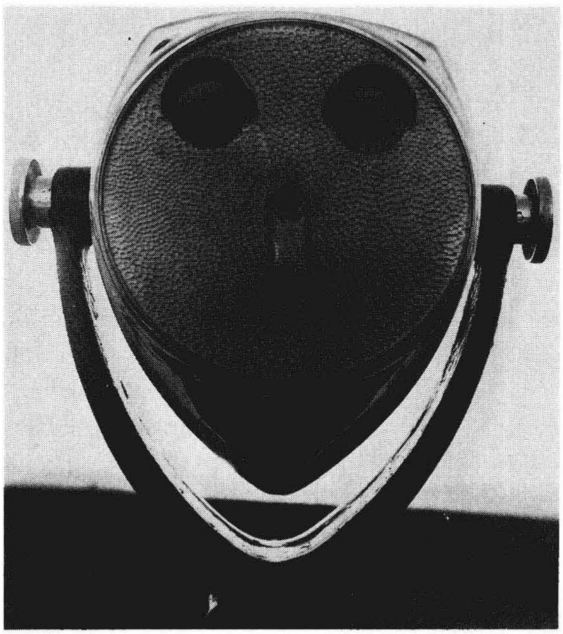
Example Mary will go if John pays her way.

p: Mary will go.
q: John will pay Mary's way.

Solution The key word is "if." The sentence can be restructured into an equivalent form:

If John pays Mary's way, then she will go.

In this restructured form, the solution becomes easy: $q \to p$.



To see is not to know or to understand. The scientist may observe various physical phenomena, but only through the use of his reasoning powers will he gain deeper insights into the real world.

Whenever a proposition is in the form of "if . . . , then . . . ," the symbolic form will be ("if" part) $\rightarrow$ ("then" part).

Example You will fail if you don't study.

> *p:* You will fail.
> *q:* You will study.

Solution As in the previous example, restructuring the sentence will make the solution easy. Rearrange the sentence, being careful not to alter its meaning.

> If you don't study, then you will fail.
> $\sim q \rightarrow p$.

Example If you eat, drink, and get merry, tomorrow you will sleep late or die.

> *p:* You eat.
> *q:* You drink.
> *r:* You get merry.
> *s:* You will sleep late tomorrow.
> *t:* You will die tomorrow.

Solution Don't be overwhelmed by complexity. Take this problem one step at a time. First consider the statement in the general form of if . . . , then . . . . This leads to the semi-symbolic form:

> (You eat, drink, and get merry) $\rightarrow$ (tomorrow you will sleep late or die).

Now take each of the two components separately:

> "You eat, drink, and get merry" can be rewritten as "You eat *and* you drink *and* you get merry" which becomes $p \wedge q \wedge r$.

The second component, "tomorrow you will sleep late or die" can be rewritten as "You will sleep late tomorrow *or* you will die tomorrow" which becomes $s \vee t$. Now put it all together.

> $(p \wedge q \wedge r) \rightarrow (s \vee t)$

Example All gamblers are wealthy.

> *p:* A given person is a gambler.
> *q:* A given person is wealthy.

Solution An equivalent form of the original proposition is:

> "If a given person is a gambler, then that person is wealthy." This new form leads to the symbolic statement $p \rightarrow q$. Note that the original proposition does not *equate* gambling and wealth. By the original proposition it is possible for some nongamblers to be wealthy. We cannot conclude that all wealthy people are gamblers.

Example Translate $\sim p \rightarrow \sim q$ into English where p and q are defined as follows:

p: You will waste.
q: You will want.

Solution First consider the $\rightarrow$, which calls for an "if . . . , then . . ." statement. Now take the negation of both components: "If you do not waste, then you will not want." We can become poetic and write simply:

"Waste not, want not."

Example Translate $(p \wedge \sim q) \rightarrow \sim r$.

p: You are willing.
q: You are able.
r: You can succeed.

Solution To get started, take the $\rightarrow$ into account by writing "if $(p \wedge \sim q)$ then $(\sim r)$." Now, $p \wedge \sim q$ means *p and not q* which means "you are willing and you are not able." $\sim r$ means "you cannot succeed." Putting it all together we get:

"If you are willing and you are not able, then you cannot succeed."

Having been exposed to some basics of symbolic logic, the student might now be questioning the relevance of that discipline to the practical world. Experience has taught that practical situations often involve vague, unreliable, or erroneous data, and rigid formal logic is not adaptable to vague, unreliable, or erroneous data. Common sense and intuition seem to operate best under those circumstances. Yet a study of formal logic will tend to improve and sharpen the tools of common sense and intuition. While logic may not help directly through a rigid and clear-cut reasoning process, it can help indirectly by better organizing the intuitive process and making it more efficient. Besides being essential to mathematical developments, logic is helpful in ordinary daily activities.

Exercises—A

In exercises 1–13, determine which of the given statements are *propositions*.

1. All numbers are evenly divisible by two.
2. Today is February 30.
3. The sum of the angles of any triangle is 180°.
4. It gets 20 miles per gallon.
5. $5 + 9 = 14$.
6. All smokers get cancer.
7. Let me count the ways.
8. $2x + 3 = 7$.
9. It's logical!
10. Millard Fillmore was a good President.

11. The World Trade Center is the tallest building in the world.
12. Now is the time for all good men to come to the aid of their party.
13. Was Lincoln the first President?

In exercises 14–19, write the appropriate symbolic form of the given proposition.

14. Janet will not drive
    p: Janet will drive.
15. If Herman returns before seven o'clock, he is on time.
    p: Herman returns before seven o'clock.
    q: Herman is on time.
16. Bart is betting two dollars, and I am not remaining in the game.
    p: Bart is betting two dollars.
    q: I am remaining in the game.
17. Ellen will fly to Los Angeles or she will take a train.
    p: Ellen will fly to Los Angeles.
    q: Ellen will take a train to Los Angeles.
18. If John does not study, then he will not pass.
    p: John studies.
    q: John will pass.
19. I will stand on my head and spit wooden nickles.
    p: I will stand on my head.
    q: I will spit wooden nickles.

In exercises 20–24, translate the given symbolic form into words by letting p and q be defined as follows.

p: You will study diligently.
q: You will succeed.

20. $p \wedge q$
21. $p \rightarrow \sim q$
22. $p \vee \sim q$
23. $\sim p \wedge \sim q$
24. $q \rightarrow \sim p$

Exercises – B

25. Is the following statement a proposition?
    "This sentence is false."

In exercises 26–29, write the appropriate symbolic form of the given proposition.

26. If I buy a new car, then I must get a job or take out a loan.
    p: I will buy a new car.
    q: I will get a job.
    r: I will take out a loan.
27. You will stay fit if you exercise.
    p: You will stay fit.
    q: You exercise.
28. If you are married, over 30, have two or more children, and earn less than $3000 annually, you will qualify for state or federal aid.
    p: You are married.
    q: You are over 30.
    r: You have two or more children.
    s: You earn less than $3000 annually.
    t: You qualify for state aid.
    u: You qualify for federal aid.
29. I can go only if I am ready, or you can go only if it is Tuesday.
    p: I can go.
    q: I am ready.
    r: You can go.
    s: It is Tuesday.

In exercises 30, 31, and 32, write the symbolic form of the given compound proposition. Letter each component of the proposition in a way that allows easy identification. For example, the letter c might be suitable for the proposition "A person is confined in a hospital." (The letters t and f should not be used since they are reserved for propositions known to be true or false.)

30. Health insurance. If a person is confined in a hospital, extended care facility or other institution for care or treatment or is confined at home under care of a physician or surgeon because of a disabling sickness or injury on the effective date of that person's coverage, no benefits are payable on account of such person until that person is no longer confined in a hospital, extended care facility or other institution, or at home.