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One | and Several | Variables

CALCULUS

ONE AND SEVERAL VARIABLES

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CALCULUS

PREFACE

Preliminary versions of this text have been used in a three semester calculus sequence at the University of Connecticut:

Chapters 1–9, plus excerpts from 10, during the first two semesters;
Chapters 12–15, plus excerpts from 10 and 11, during the third semester.

We have deliberately avoided the exotic and stayed within the main stream of calculus. We have done this in the belief that the so-called standard material is the most important, the most useful, and the most interesting. What we have tried to do here is present the standard material in a congenial manner.

S.L.S.
E.H.

ACKNOWLEDGMENTS

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THE GREEK ALPHABET

A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ ϑ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π ϖ	pi
P	ρ	rho
Σ	σ ς	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ φ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

CALCULUS

This book is available both as a complete volume and in two parts. The complete volume is intended for three semesters. In the two-part version, Part I can be used for a two-semester course and deals with functions of one variable, analytic geometry, and infinite series (Chapters 1–11 of the complete volume). Part II contains infinite series, vectors, and functions of several variables (Chapters 10–15 of the complete volume).

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7 SOME ANALYTIC GEOMETRY

8 THE TECHNIQUE OF INTEGRATION

9 INTEGRATION AS AN AVERAGING PROCESS; APPLICATIONS

10 SEQUENCES AND SERIES

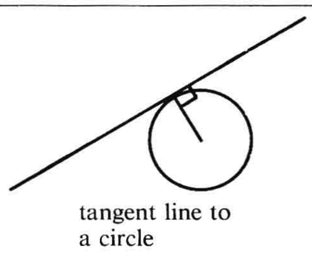
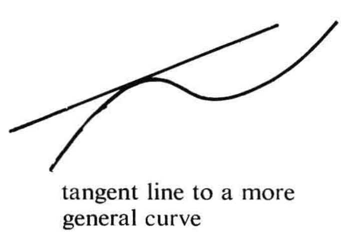
1

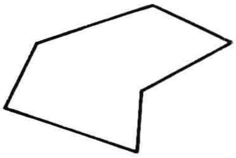
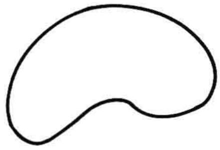
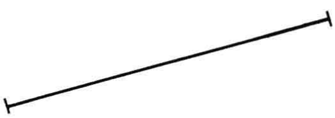
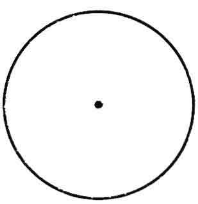
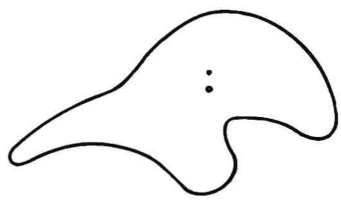
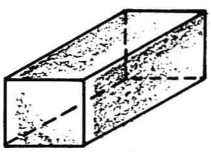
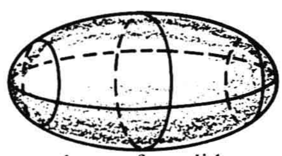
INTRODUCTION

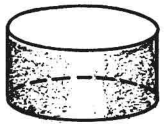
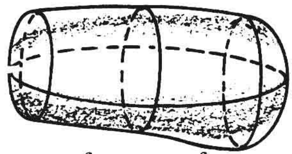
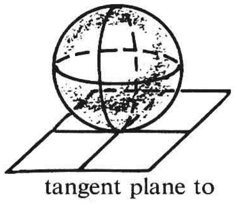
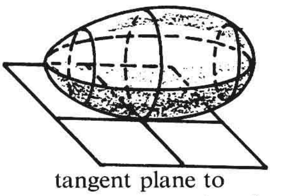
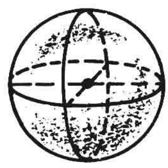
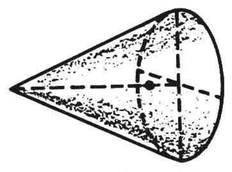
1.1 What is Calculus?

To a Roman in the days of the empire a "calculus" was a little pebble that he used in counting and in gambling. Centuries later the verb "calculare" came to mean "to compute," "to reckon," "to figure out." To the engineer and mathematician of today calculus is the branch of mathematics that takes in elementary algebra and geometry and adds one more ingredient: *the limit process.*

Calculus begins where elementary mathematics leaves off. It takes ideas from elementary mathematics and extends them to a much more general situation. Here are some examples. On the left-hand side you will find an idea from elementary mathematics; on the right, this same idea as enriched by calculus.

Elementary Mathematics	*Calculus*
 slope of a line $y = mx + b$	 slope of a curve $y = f(x)$
 tangent line to a circle	 tangent line to a more general curve

average velocity, average acceleration	instantaneous velocity, instantaneous acceleration
distance moved under a constant velocity	distance moved under varying velocity
 area of a region bounded by line segments	 area of a region bounded by curves
sum of a finite collection of numbers $a_1 + a_2 + \cdots + a_n$	sum of an infinite series $a_1 + a_2 + \cdots + a_n + \cdots$
average of a finite collection of numbers	average value of a function on an interval
 length of a line segment	 length of a curve
 center of a circle	 center of gravity of a region
 volume of a rectangular solid	 volume of a solid with a curved boundary

 surface area of a cylinder	 surface area of a more general solid
 tangent plane to a sphere	 tangent plane to a more general surface
work done by a constant force	work done by a varying force
mass of an object of constant density	mass of an object of varying density
 center of a sphere	 center of gravity of a more general solid

1.2 Notions and formulas from elementary mathematics

The following outline is presented for review and easy reference.

I Sets

the object x is in the set A: $x \in A$
(x is an element of A)
the object x is not in the set A: $x \notin A$
containment: $A \subseteq B$
union: $A \cup B$
intersection: $A \cap B$
empty set: a set with no elements

(These are the only notions from set theory that you will need for this book. If you are not familiar with them, see Section 1 of the appendix.)

II Real Numbers

(1) CLASSIFICATION

integers: 0, 1, −1, 2, −2, 3, −3, etc.

rational numbers: $\dfrac{p}{q}$ where p and q are integers, $q \neq 0$

irrational numbers: real numbers which are not rational; for example, $\sqrt{2}$, π.

(2) ORDER PROPERTIES

(i) $a < b$, $b < a$, or $a = b$
(ii) if $a < b$ and $b < c$, then $a < c$
(iii) if $a < b$, then $a + c < b + c$ for all real numbers c
(iv) if $a < b$ and $c > 0$, then $ac < bc$
if $a < b$ and $c < 0$, then $ac > bc$

(3) DENSITY

Between any two numbers there is a rational number and an irrational number.

(4) ABSOLUTE VALUE

$$|a| = \sqrt{a^2};\ |a| = \max\{a, -a\};\ |a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a \leq 0 \end{cases}$$

geometric interpretation: $|a|$ = distance between a and 0
$|a - c|$ = distance between a and c

properties: (i) $|a| = 0$ iff $a = 0$†
(ii) $|-a| = |a|$
(iii) $|ab| = |a|\,|b|$
(iv) $|a + b| \leq |a| + |b|$ (the triangle inequality)
(v) $\big||a| - |b|\big| \leq |a - b|$ (alternate form of the triangle inequality)

(5) INTERVALS

The notation

$$\{x\colon (\quad)\}$$

is used to denote the set of all x such that ( ). Suppose now that $a < b$. The *open interval* (a, b) is the set of all numbers between a and b:

$$(a, b) = \{x\colon a < x < b\}.$$

The *closed interval* $[a, b]$ is the open interval (a, b) plus the end points

$$[a, b] = \{x\colon a \leq x \leq b\}.$$

† By "iff" we mean "if and only if." This expression is used so often in mathematics that it is convenient to have an abbreviation for it.

There are seven other types of intervals:

$$(a, b] = \{x: a < x \leq b\}, \qquad [a, b) = \{x: a \leq x < b\},$$
$$(a, \infty) = \{x: a < x\}, \qquad [a, \infty) = \{x: a \leq x\},$$
$$(-\infty, b) = \{x: x < b\}, \qquad (-\infty, b] = \{x: x \leq b\},$$
$$(-\infty, \infty) = \mathbf{R} = \text{set of real numbers}$$

(6) BOUNDEDNESS

A set S of real numbers is said to be

(i) *bounded above* iff there exists a real number M such that

$$x \leq M \qquad \text{for all} \quad x \in S.$$

(M is called an *upper bound* for S)

(ii) *bounded below* iff there exists a real number m such that

$$m \leq x \qquad \text{for all} \quad x \in S.$$

(m is called a *lower bound* for S)

(iii) *bounded* iff it is bounded above and below.

Example. The intervals $(-\infty, 2]$ and $(-\infty, 2)$ are bounded above but not below.

Example. The set of positive integers is bounded below but not above.

Example. The intervals $[0, 1]$, $(0, 1)$, and $(0, 1]$ are bounded (both above and below).

III Algebra and Geometry

(1) GENERAL QUADRATIC FORMULA

The quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(2) FACTORIALS

$$0! = 1$$
$$1! = 1$$
$$2! = 2 \cdot 1$$
$$3! = 3 \cdot 2 \cdot 1$$
$$\vdots$$
$$n! = n(n-1) \cdots (2)(1).$$