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Preface

This book is devoted to C^* -algebras as a tool in numerical analysis. Some readers might consider the use of C^* -algebras to study properties of approximation methods as unusual and exotic. We would like to encourage them to read and see for themselves the power of such techniques both for the investigation of very concrete discretization procedures and for establishing the theoretical foundation of numerical analysis. For a general overview of the fruitful interplay between C^* -techniques, concrete operator theory, and numerical analysis and, thus, of the contents of this book, we refer the reader to the Introduction.

The book is addressed to a wide audience. We hope that it proves to be of use both for the student who wants to see applications of functional analysis and to learn numerical analysis, and for the mathematician and the engineer interested in theoretical aspects of numerical analysis.

We wish to express our sincere appreciation to our friends and colleagues, Albrecht Böttcher, Torsten Ehrhardt, Peter Junghanns, and Marko Lindner, who read the bulk of the manuscript very carefully and not only made many corrections but also offered constructive criticism to improve the book substantially. Our students, Michael Ehrenberger and Florian Meyer, did an excellent job in drawing the figures and performing the test calculations. One of the authors (S. R.) was supported by a DFG Heisenberg grant while working on the manuscript. He is grateful to the German Research Foundation for this support, as well as to Bernd Kirstein and Wolfgang Wendland and their staffs for their hospitality during that time. Finally, we are pleased to express our gratitude to the publisher, Marcel Dekker, Inc., and to the mathematics series editor, Prof. Zuhair Nashed, for inviting us to write this monograph and for their careful work on the book.

*Roland Hagen
Steffen Roch
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Contents

Preface	3
0 Introduction	11
0.1 Numerical analysis	11
0.2 Operator chemistry	14
0.3 The algebraic language of numerical analysis	15
0.4 Microscoping	18
0.5 A few remarks on economy	21
0.6 Brief description of the contents	22
1 The algebraic language of numerical analysis	25
1.1 Approximation methods	25
1.1.1 Basic definitions	26
1.1.2 Projection methods	28
1.1.3 Finite section method	31
1.2 Banach algebras and stability	34
1.2.1 Algebras, ideals and homomorphisms	35
1.2.2 Algebraization of stability	36
1.2.3 Small perturbations	39
1.2.4 Compact perturbations	39
1.3 Finite sections of Toeplitz operators with continuous generating function	45
1.3.1 Laurent, Toeplitz and Hankel operators	45
1.3.2 Invertibility and Fredholmness of Toeplitz operators	48
1.3.3 The finite section method	49
1.4 C^* -algebras of approximation sequences	52
1.4.1 C^* -algebras, their ideals and homomorphisms	53
1.4.2 The Toeplitz C^* -algebra and the C^* -algebra of the finite section method for Toeplitz operators	56

1.4.3	Stability of sequences in the C^* -algebra of the finite section method for Toeplitz operators	60
1.4.4	Symbol of the finite section method for Toeplitz operators	61
1.5	Asymptotic behaviour of condition numbers	62
1.5.1	The condition of an operator	63
1.5.2	Convergence of norms	64
1.5.3	Condition numbers of finite sections of Toeplitz operators	65
1.6	Fractality of approximation methods	66
1.6.1	Fractal homomorphisms, fractal algebras, fractal sequences	67
1.6.2	Fractal algebras, and convergence of norms	71
	Notes and references	73
2	Regularization of approximation methods	75
2.1	Stably regularizable sequences	76
2.1.1	Moore-Penrose inverses and regularizations of matrices	76
2.1.2	Moore-Penrose inverses and regularization of operators	80
2.1.3	Stably regularizable approximation sequences	85
2.2	Algebraic characterization of stably regularizable sequences	89
2.2.1	Moore-Penrose invertibility in C^* -algebras	89
2.2.2	Stable regularizability, and Moore-Penrose invertibility in $\mathcal{F}/\mathcal{G}$	92
2.2.3	Finite sections of Toeplitz operators and their stable regularizability	97
2.2.4	Convergence of generalized condition numbers	100
2.2.5	Difficulties with Moore-Penrose stability	103
	Notes and references	104
3	Approximation of spectra	105
3.1	Set sequences	105
3.1.1	Limiting sets of set functions	106
3.1.2	Coincidence of the partial and uniform limiting set	108
3.2	Spectra and their limiting sets	110
3.2.1	Limiting sets of spectra of norm convergent sequences	112
3.2.2	Limiting sets of spectra: the general case	114
3.2.3	The case of fractal sequences	117
3.2.4	Limiting sets of singular values	119

3.3	Pseudospectra and their limiting sets	119
3.3.1	ε -invertibility	119
3.3.2	Limiting sets of pseudospectra	125
3.3.3	The case of fractal sequences	127
3.3.4	Pseudospectra of operator polynomials	128
3.4	Numerical ranges and their limiting sets	134
3.4.1	Spatial and algebraic numerical ranges	134
3.4.2	Limiting sets of numerical ranges	136
3.4.3	The case of fractal sequences	140
	Notes and references	143
4	Stability analysis for concrete approximation methods	145
4.1	Local principles	146
4.1.1	Commutative C^* -algebras	146
4.1.2	The local principle by Allan and Douglas	149
4.1.3	Fredholmness of Toeplitz operators with piecewise continuous generating function	151
4.2	Finite sections of Toeplitz operators generated by a piecewise continuous function	158
4.2.1	The lifting theorem	158
4.2.2	Application of the local principle	163
4.2.3	Galerkin methods with spline ansatz for singular in- tegral equations	167
4.3	Finite sections of Toeplitz operators generated by a quasi- continuous function	169
4.3.1	Quasicontinuous functions	169
4.3.2	Stability of the finite section method	173
4.3.3	Some other classes of oscillating functions	175
4.4	Polynomial collocation methods for singular integral opera- tors with piecewise continuous coefficients	177
4.4.1	Singular integral operators	178
4.4.2	Stability of the polynomial collocation method	183
4.4.3	Collocation versus Galerkin methods	187
4.5	Paired circulants and spline approximation methods	188
4.5.1	Circulants and paired circulants	190
4.5.2	The stability theorem	191
4.6	Finite sections of band-dominated operators	197
4.6.1	Multidimensional band dominated operators	197
4.6.2	Fredholmness of band dominated operators	198
4.6.3	Finite sections of band dominated operators	200
	Notes and references	204

5	Representation theory	207
5.1	Representations	208
5.1.1	The spectrum of a C^* -algebra	208
5.1.2	Primitive ideals	210
5.1.3	The spectrum of an ideal and of a quotient	212
5.1.4	Representations of some concrete algebras	213
5.2	Postliminal algebras	222
5.2.1	Liminal and postliminal algebras	223
5.2.2	Dual algebras	226
5.2.3	Finite sections of Wiener-Hopf operators with almost periodic generating function	230
5.3	Lifting theorems and representation theory	238
5.3.1	Lifting one ideal	238
5.3.2	The lifting theorem	239
5.3.3	Sufficient families of homomorphisms	243
5.3.4	Structure of fractal lifting homomorphisms	249
	Notes and references	254
6	Fredholm sequences	255
6.1	Fredholm sequences in standard algebras	256
6.1.1	The standard model	256
6.1.2	Fredholm sequences	258
6.1.3	Fredholm sequences and stable regularizability	259
6.1.4	Fredholm sequences and Moore-Penrose stability	260
6.2	Fredholm sequences and the asymptotic behavior of singular values	264
6.2.1	The main result	265
6.2.2	A distinguished element and its range dimension	266
6.2.3	Upper estimate of $\dim \operatorname{Im} \Pi_n$	269
6.2.4	Lower estimate of $\dim \operatorname{Im} \Pi_n$	270
6.2.5	Some examples	276
6.3	A general Fredholm theory	282
6.3.1	Centrally compact and Fredholm sequences	282
6.3.2	Fredholmness modulo compact elements	288
6.3.3	Fredholm sequences in standard algebras	297
6.4	Weakly Fredholm sequences	305
6.4.1	Sequences with finite splitting property	305
6.4.2	Properties of weakly Fredholm sequences	305
6.4.3	Strong limits of weakly Fredholm sequences	307
6.4.4	Weakly Fredholm sequences of matrices	313
6.5	Some applications	314
6.5.1	Numerical determination of the kernel dimension.	314

6.5.2	Around the finite section method for Toeplitz operators	315
6.5.3	Discretization of shift operators	317
	Notes and references	322
7	Self-adjoint approximation sequences	323
7.1	The spectrum of a self-adjoint approximation sequence . . .	323
7.1.1	Essential and transient points	323
7.1.2	Fractality of self-adjoint sequences	327
7.1.3	Arveson dichotomy: band operators	333
7.1.4	Arveson dichotomy: standard algebras	338
7.2	Szegő-type theorems	339
7.2.1	Følner and Szegő algebras	340
7.2.2	Szegő's theorem revisited	346
7.2.3	A further generalization of Szegő's theorem	348
7.2.4	Algebras with unique tracial state	352
	Notes and references	354
	Bibliography	357
	Index	373