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John C. Oxtoby

Measure and Category

A Survey of the Analogies between Topological and Measure Spaces

Second Edition

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Preface to the Second Edition

In this edition, a set of Supplementary Notes and Remarks has been added at the end, grouped according to chapter. Some of these call attention to subsequent developments, others add further explanation or additional remarks. Most of the remarks are accompanied by a briefly indicated proof, which is sometimes different from the one given in the reference cited. The list of references has been expanded to include many recent contributions, but it is still not intended to be exhaustive.

Bryn Mawr, April 1980 John C. Oxtoby

Preface to the First Edition

This book has two main themes: the Baire category theorem as a method for proving existence, and the "duality" between measure and category. The category method is illustrated by a variety of typical applications, and the analogy between measure and category is explored in all of its ramifications. To this end, the elements of metric topology are reviewed and the principal properties of Lebesgue measure are derived. It turns out that Lebesgue integration is not essential for present purposes—the Riemann integral is sufficient. Concepts of general measure theory and topology are introduced, but not just for the sake of generality. Needless to say, the term "category" refers always to Baire category; it has nothing to do with the term as it is used in homological algebra.

A knowledge of calculus is presupposed, and some familiarity with the algebra of sets. The questions discussed are ones that lend themselves naturally to set-theoretical formulation. The book is intended as an introduction to this kind of analysis. It could be used to supplement a standard course in real analysis, as the basis for a seminar, or for independent study. It is primarily expository, but a few refinements of known results are included, notably Theorem 15.6 and Proposition 20.4. The references are not intended to be complete. Frequently a secondary source is cited where additional references may be found.

The book is a revised and expanded version of notes originally prepared for a course of lectures given at Haverford College during the spring of 1957 under the auspices of the William Pyle Philips Fund. These, in turn, were based on the Earle Raymond Hedrick Lectures presented at the Summer Meeting of the Mathematical Association of America at Seattle, Washington, in August, 1956.

Bryn Mawr, April 1971 John C. Oxtoby

Contents

1. Measure and Category on the Line

The notions of measure and category are based on that of countability. Cantor's theorem, which says that no interval of real numbers is countable, provides a natural starting point for the study of both measure and category. Let us recall that a set is called *denumerable* if its elements can be put in one-to-one correspondence with the natural numbers $1, 2, \ldots$. A *countable* set is one that is either finite or denumerable. The set of rational numbers is denumerable, because for each positive integer k there are only a finite number ($\leq 2k-1$) of rational numbers p/q in reduced form ($q>0$, p and q relatively prime) for which $|p|+q=k$. By numbering those for which $k=1$, then those for which $k=2$, and so on, we obtain a sequence in which each rational number appears once and only once. Cantor's theorem reads as follows.

Theorem 1.1 (Cantor). *For any sequence $\{a_n\}$ of real numbers and for any interval I there exists a point p in I such that $p \neq a_n$ for every n.*

One proof runs as follows. Let I_1 be a closed subinterval of I such that $a_1 \notin I_1$. Let I_2 be a closed subinterval of I_1 such that $a_2 \notin I_2$. Proceeding inductively, let I_n be a closed subinterval of I_{n-1} such that $a_n \notin I_n$. The nested sequence of closed intervals I_n has a non-empty intersection. If $p \in \bigcap I_n$, then $p \in I$ and $p \neq a_n$ for every n.

This proof involves infinitely many unspecified choices. To avoid this objection the intervals must be chosen according to some definite rule. One such rule is this: divide I_{n-1} into three subintervals of equal length and take for I_n the first one of these that does not contain a_n. If we take I_0 to be the closed interval concentric with I and half as long, say, then all the choices are specified, and we have a well defined function of $(I, a_1, a_2, \ldots)$ whose value is a point of I different from all the a_n.

The fact that no interval is countable is an immediate corollary of Cantor's theorem.

With only a few changes, the above proof becomes a proof of the Baire category theorem for the line. Before we can formulate this theorem we need some definitions. A set A is *dense in the interval I* if A has a non-empty intersection with every subinterval of I; it is called *dense* if it is dense in the line R. A set A is *nowhere dense* if it is dense in no interval, that is, if every interval has a subinterval contained in the complement of A. A nowhere dense set may be characterized as one that is "full of holes." The definition can be stated in two other useful ways: A is nowhere dense if and only if its complement A' contains a dense open set, and if and only if $\bar{A}$ (or A^-, the closure of A) has no interior points. The class of nowhere dense sets is closed under certain operations, namely

Theorem 1.2. *Any subset of a nowhere dense set is nowhere dense. The union of two (or any finite number) of nowhere dense sets is nowhere dense. The closure of a nowhere dense set is nowhere dense.*

Proof. The first statement is obvious. To prove the second, note that if A_1 and A_2 are nowhere dense, then for each interval I there is an interval $I_1 \subset I - A_1$ and an interval $I_2 \subset I_1 - A_2$. Hence $I_2 \subset I - (A_1 \cup A_2)$. This shows that $A_1 \cup A_2$ is nowhere dense. Finally, any open interval contained in A' is also contained in $A^{-\prime}$. □

A denumerable union of nowhere dense sets is not in general nowhere dense, it may even be dense. For instance, the set of rational numbers is dense, but it is also a denumerable union of singletons (sets having just one element), and singletons are nowhere dense in R.

A set is said to be of *first category* if it can be represented as a countable union of nowhere dense sets. A subset of R that cannot be so represented is said to be of *second category*. These definitions were formulated in 1899 by R. Baire [18, p. 48], to whom the following theorem is due.

Theorem 1.3 (Baire). *The complement of any set of first category on the line is dense. No interval in R is of first category. The intersection of any sequence of dense open sets is dense.*

Proof. The three statements are essentially equivalent. To prove the first, let $A = \bigcup A_n$ be a representation of A as a countable union of nowhere dense sets. For any interval I, let I_1 be a closed subinterval of $I - A_1$. Let I_2 be a closed subinterval of $I_1 - A_2$, and so on. Then $\bigcap I_n$ is a non-empty subset of $I - A$, hence A' is dense. To specify all the choices in advance, it suffices to arrange the (denumerable) class of closed intervals with rational endpoints into a sequence, take $I_0 = I$, and for $n > 0$ take I_n to be the first term of the sequence that is contained in $I_{n-1} - A_n$.

The second statement is an immediate corollary of the first. The third statement follows from the first by complementation. □

Evidently Baire's theorem implies Cantor's. Its proof is similar, although a different rule for choosing I_n was needed.

Theorem 1.4. *Any subset of a set of first category is of first category. The union of any countable family of first category sets is of first category.*

It is obvious that the class of first category sets has these closure properties. However, the closure of a set of first category is not in general of first category. In fact, the closure of a linear set A is of first category if and only if A is nowhere dense.

A class of sets that contains countable unions and arbitrary subsets of its members is called a σ-ideal. The class of sets of first category and the class of countable sets are two examples of σ-ideals of subsets of the line. Another example is the class of nullsets, which we shall now define.

The length of any interval I is denoted by $|I|$. A set $A \subset R$ is called a *nullset* (or a set of *measure zero*) if for each $\varepsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \bigcup I_n$ and $\sum |I_n| < \varepsilon$.

It is obvious that singletons are nullsets and that any subset of a nullset is a nullset. Any countable union of nullsets is also a nullset. For suppose A_i is a nullset for $i = 1, 2, \ldots$. Then for each i there is a sequence of intervals I_{ij} $(j = 1, 2, \ldots)$ such that $A_i \subset \bigcup_j I_{ij}$ and $\sum_j |I_{ij}| < \varepsilon/2^i$. The set of all the intervals I_{ij} covers A, and $\sum_{ij} |I_{ij}| < \varepsilon$, hence A is a nullset. This shows that the class of nullsets is a σ-ideal. Like the class of sets of first category, it includes all countable sets.

Theorem 1.5 (Borel). *If a finite or infinite sequence of intervals I_n covers an interval I, then $\sum |I_n| \geqq |I|$.*

Proof. Assume first that $I = [a, b]$ is closed and that all of the intervals I_n are open. Let (a_1, b_1) be the first interval that contains a. If $b_1 \leqq b$, let (a_2, b_2) be the first interval of the sequence that contains b_1. If $b_{n-1} \leqq b$, let (a_n, b_n) be the first interval that contains b_{n-1}. This procedure must terminate with some $b_N > b$. Otherwise the increasing sequence $\{b_n\}$ would converge to a limit $x \leqq b$, and x would belong to I_k for some k. All but a finite number of the intervals (a_n, b_n) would have to precede I_k in the given sequence, namely, all those for which $b_{n-1} \in I_k$. This is impossible, since no two of these intervals are equal. (Incidentally, this reasoning reproduces Borel's own proof of the "Heine-Borel theorem" [5, p. 228].) We have

$$b - a < b_N - a_1 = \textstyle\sum_{i=2}^{N} (b_i - b_{i-1}) + b_1 - a_1 \leqq \sum_{i=1}^{N} (b_i - a_i),$$

and so the theorem is true in this case.

In the general case, for any $\alpha > 1$ let J be a closed subinterval of I with $|J| = |I|/\alpha$, and let J_n be an open interval containing I_n with $|J_n| = \alpha |I_n|$. Then J is covered by the sequence $\{J_n\}$. We have already shown that

$\sum|J_n| \geqq |J|$. Hence $\alpha\sum|I_n| = \sum|J_n| \geqq |J| = |I|/\alpha$. Letting $\alpha \to 1$ we obtain the desired conclusion. □

This theorem implies that no interval is a nullset; it therefore provides still another proof of Cantor's theorem.

Every countable set is of first category and of measure zero. Some uncountable sets also belong to both classes. The simplest example is the *Cantor set* C, which consists of all numbers in the interval $[0, 1]$ that admit a ternary development in which the digit 1 does not appear. It can be constructed by deleting the open middle third of the interval $[0, 1]$, then deleting the open middle thirds of each of the intervals $[0, 1/3]$ and $[2/3, 1]$, and so on. If F_n denotes the union of the 2^n closed intervals of length $1/3^n$ which remain at the n-th stage, then $C = \bigcap F_n$. C is closed, since it is an intersection of closed sets. C is nowhere dense, since F_n (and therefore C) contains no interval of length greater than $1/3^n$. The sum of the lengths of the intervals that compose F_n is $(2/3)^n$, which is less than ε if n is taken sufficiently large. Hence C is a nullset. Finally, each number x in $(0, 1]$ has a unique non-terminating binary development $x = .x_1x_2x_3\ldots$. If $y_i = 2x_i$, then $.y_1y_2y_3\ldots$ is the ternary development with $y_i \neq 1$ of some point y of C. This correspondence between x and y, extended by mapping 0 onto itself, defines a one-to-one map of $[0, 1]$ onto a (proper) subset of C. It follows that C is uncountable; it has cardinality c (the power of the continuum).

The sets of measure zero and the sets of first category constitute two σ-ideals, each of which properly contains the class of countable sets. Their properties suggest that a set belonging to either class is "small" in one sense or another. A nowhere dense set is small in the intuitive geometric sense of being perforated with holes, and a set of first category can be "approximated" by such a set. A set of first category may or may not have any holes, but it always has a dense set of gaps. No interval can be represented as the union of a sequence of such sets. On the other hand, a nullset is small in the metric sense that it can be covered by a sequence of intervals of arbitrarily small total length. If a point is chosen at random in an interval in such a way that the probability of its belonging to any subinterval J is proportional to $|J|$, then the probability of its belonging to any given nullset is zero. It is natural to ask whether these notions of smallness are related. Does either class include the other? That neither class does, and that in some cases the two notions may be diametrically opposed, is shown by the following

Theorem 1.6. *The line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.*

Proof. Let $a_1, a_2, \ldots$ be an enumeration of the set of rational numbers (or of any countable dense subset of the line). Let I_{ij} be the open interval with center a_i and length $1/2^{i+j}$. Let $G_j = \bigcup_{i=1}^{\infty} I_{ij}$ ($j = 1, 2, \ldots$) and $B = \bigcap_{j=1}^{\infty} G_j$. For any $\varepsilon > 0$ we can choose j so that $1/2^j < \varepsilon$. Then $B \subset \bigcup_i I_{ij}$ and $\sum_i |I_{ij}| = \sum_i 1/2^{i+j} = 1/2^j < \varepsilon$. Hence B is a nullset. On the other hand, G_j is a dense open subset of R, since it is the union of a sequence of open intervals and it includes all rational points. Therefore its complement G_j' is nowhere dense, and $A = B' = \bigcup_j G_j'$ is of first category. □

Corollary 1.7. *Every subset of the line can be represented as the union of a nullset and a set of first category.*

There is of course nothing paradoxical in the fact that a set that is small in one sense may be large in some other sense.

2. Liouville Numbers

Cantor's theorem, Baire's theorem, and Borel's theorem are existence theorems. If one can show that the set of numbers in an interval that lack a certain property is either countable, or a nullset, or a set of first category, then it follows that there exist points of the interval that have the property in question, in fact, most points of the interval (in the sense of cardinal number, or measure, or category, respectively) have the property. As a first illustration of this method let us consider the existence of transcendental numbers.

A complex number z is called *algebraic* if it satisfies some equation of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

with integer coefficients, not all zero. The *degree* of an algebraic number z is the smallest positive integer n such that z satisfies an equation of degree n. For instance, any rational number is algebraic of degree 1, $\sqrt{2}$ is algebraic of degree 2, and $\sqrt{2} + \sqrt{3}$ is algebraic of degree 4. Any real number that is not algebraic is called *transcendental.* Do there exist transcendental numbers? In view of Cantor's theorem, this question is answered by the following

Theorem 2.1. *The set of real algebraic numbers is denumerable.*

Proof. Let us define the weight of a polynomial $f(x) = \sum_0^n a_i x^i$ to be the number $n + \sum_0^n |a_i|$. There are only a finite number of polynomials having a given weight. Arrange these in some order, say lexicographically (first in order of n, then in order of a_0, and so on). Every non-constant polynomial has a weight at least equal to 2. Taking the polynomials of weight 2 in order, then those of weight 3, and so on, we obtain a sequence $f_1, f_2, f_3, \ldots$ in which every polynomial of degree 1 or more appears just once. Each polynomial has at most a finite number of real zeros. Number the zeros of f_1 in order, then those of f_2, and so on, passing over any that have already been numbered. In this way we obtain a definite enumeration of all real algebraic numbers. The sequence is infinite because it includes all rational numbers. ☐

This gives perhaps the simplest proof of the existence of transcendental numbers. It should be noted that it is not an indirect proof; when all the choices are fixed in advance the construction used to prove Theorem 1.1 defines a specific transcendental number in [0, 1]. It may be laborious to compute even a few terms of its decimal development, but in principle the number can be computed to any desired accuracy.

An older and more informative proof of the existence of transcendental numbers is due to Liouville. His proof is based on the following

Lemma 2.2. *For any real algebraic number z of degree $n > 1$ there exists a positive integer M such that*

$$\left| z - \frac{p}{q} \right| > \frac{1}{Mq^n}$$

for all integers p and q, $q > 0$.

Proof. Let $f(x)$ be a polynomial of degree n with integer coefficients for which $f(z) = 0$. Let M be a positive integer such that $|f'(x)| \leqq M$ whenever $|z - x| \leqq 1$. Then, by the mean value theorem,

$$(1) \qquad |f(x)| = |f(z) - f(x)| \leqq M|z - x| \quad \text{whenever} \quad |z - x| \leqq 1 .$$

Now consider any two integers p and q, with $q > 0$. We wish to show that $|z - p/q| > 1/Mq^n$. This is evidently true in case $|z - p/q| > 1$, so we may assume that $|z - p/q| \leqq 1$. Then, by (1), $|f(p/q)| \leqq M|z - p/q|$, and therefore

$$(2) \qquad |q^n f(p/q)| \leqq Mq^n|z - p/q| .$$

The equation $f(x) = 0$ has no rational root (otherwise z would satisfy an equation of degree less then n). Moreover, $q^n f(p/q)$ is an integer. Hence the left member of (2) is at least 1 and we infer that $|z - p/q| \geqq 1/Mq^n$. Equality cannot hold, because z is irrational. □

A real number z is called a *Liouville number* if z is irrational and has the property that for each positive integer n there exist integers p and q such that

$$\left| z - \frac{p}{q} \right| < \frac{1}{q^n} \quad \text{and} \quad q > 1 .$$

For example, $z = \sum_1^\infty 1/10^{k!}$ is a Liouville number. (Take $q = 10^{n!}$.)

Theorem 2.3. *Every Liouville number is transcendental.*

Proof. Suppose some Liouville number z is algebraic, of degree n. Then $n > 1$, since z is irrational. By Lemma 2.2 there exists a positive integer M such that

$$(3) \qquad |z - p/q| > 1/Mq^n$$

for all integers p and q with $q > 0$. Choose a positive integer k such that $2^k \geqq 2^n M$. Because z is a Liouville number there exist integers p and q, with $q > 1$, such that

(4) $$|z - p/q| < 1/q^k .$$

From (3) and (4) it follows that $1/q^k > 1/Mq^n$. Hence $M > q^{k-n} \geqq 2^{k-n} \geqq M$, a contradiction. □

Let us examine the set E of Liouville numbers. From the definition it follows at once that

(5) $$E = Q' \cap \bigcap_{n=1}^{\infty} G_n ,$$

where Q denotes the set of rational numbers and

$$G_n = \bigcup_{q=2}^{\infty} \bigcup_{p=-\infty}^{\infty} (p/q - 1/q^n, p/q + 1/q^n) .$$

G_n is a union of open intervals. Moreover, G_n includes every number of the form p/q, $q \geqq 2$, hence $G_n \supset Q$. Therefore G_n is a dense open set, and so its complement is nowhere dense. Since, by (5), $E' = Q \cup \bigcup_{n=1}^{\infty} G_n'$, it follows that E' is of first category. Thus Baire's theorem implies that Liouville transcendental numbers exist in every interval, they are the "general case" in the sense of category.

What about the measure of E? From (5) it follows that $E \subset G_n$ for every n. Let

$$G_{n,q} = \bigcup_{p=-\infty}^{\infty} (p/q - 1/q^n, p/q + 1/q^n) \qquad (q = 2, 3, \ldots)$$

For any two positive integers m and n we have

$$E \cap (-m, m) \subset G_n \cap (-m, m)$$
$$= \bigcup_{q=2}^{\infty} [G_{n,q} \cap (-m, m)] \subset \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} (p/q - 1/q^n, p/q + 1/q^n) .$$

Therefore $E \cap (-m, m)$ can be covered by a sequence of intervals the sum of whose lengths, for any $n > 2$, is

$$\sum_{q=2}^{\infty} \sum_{p=-mq}^{mq} 2/q^n = \sum_{q=2}^{\infty} (2mq + 1)(2/q^n) \leqq \sum_{q=2}^{\infty} (4mq + q)(1/q^n)$$
$$= (4m + 1) \sum_{q=2}^{\infty} 1/q^{n-1} \leqq (4m + 1) \int_1^{\infty} \frac{dx}{x^{n-1}} = \frac{4m + 1}{n - 2} .$$

It follows that $E \cap (-m, m)$ is a nullset for every m, and therefore E is a nullset.

Thus E is small in the sense of measure, but large in the sense of category. The sets E and E' provide another decomposition of the line into a set of measure zero and a set of first category (cf. Theorem 1.6). Moreover, the set E is small in an even stronger sense, as we shall now show.

If s is a positive real number and $E \subset R$, then E is said to have s*-dimensional Hausdorff measure zero* if for each $\varepsilon > 0$ there is a sequence of intervals I_n such that $E \subset \bigcup_{n=1}^{\infty} I_n$, $\sum_{n=1}^{\infty} |I_n|^s < \varepsilon$, and $|I_n| < \varepsilon$ for every n. The sets of s-dimensional measure zero constitute a σ-ideal. For $s = 1$ it coincides with the class of nullsets, and for $0 < s < 1$ it is a proper subclass. The following theorem therefore strengthens the proposition that E is a nullset.

Theorem 2.4. *The set E of Liouville numbers has s-dimensional Hausdorff measure zero, for every $s > 0$.*

Proof. It suffices to find, for each $\varepsilon > 0$ and for each positive integer m, a sequence of intervals I_n such that

$$E \cap (-m, m) \subset \bigcup_{n=1}^{\infty} I_n, \quad \sum_{n=1}^{\infty} |I_n|^s < \varepsilon, \quad \text{and} \quad |I_n| < \varepsilon .$$

For each positive integer n, we have

$$E \cap (-m, m) \subset \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} (p/q - 1/q^n, p/q + 1/q^n) .$$

Choose n so as to satisfy simultaneously the following conditions:

$$1/2^{n-1} < \varepsilon , \qquad ns > 2 , \qquad \frac{(2m+1)\, 2^s}{ns - 2} < \varepsilon .$$

Then each of the intervals $(p/q - 1/q^n, p/q + 1/q^n)$ has length $2/q^n \leqq 2/2^n < \varepsilon$, and we have

$$\sum_{q=2}^{\infty} \sum_{p=-mq}^{mq} (2/q^n)^s = \sum_{q=2}^{\infty} \frac{(2mq+1)\, 2^s}{q^{ns}}$$

$$\leqq (2m+1)\, 2^s \sum_{q=2}^{\infty} \frac{1}{q^{ns-1}} \leqq (2m+1)\, 2^s \int_1^{\infty} \frac{dx}{x^{ns-1}}$$

$$= \frac{(2m+1)\, 2^s}{ns-2} < \varepsilon . \quad \square$$

3. Lebesgue Measure in r-Space

By an *interval* I in Euclidean r-space ($r=1, 2, \ldots$) is meant a rectangular parallelepiped with edges parallel to the axes. It is the Cartesian product of r 1-dimensional intervals. As in the 1-dimensional case, the r-dimensional volume of I will be denoted by $|I|$. Lebesgue measure in r-space is an extension of the notion of volume to a larger class of sets. Thus Lebesgue measure has a different meaning in spaces of different dimension. However, since we shall usually regard the dimension as fixed, there is no need to indicate r explicitly in our notations.

A sequence of intervals I_i is said to *cover* the set A if its union contains A. The greatest lower bound of the sums $\sum |I_i|$, for all sequences $\{I_i\}$ that cover A, is called the *outer measure* of A; it is denoted by $m^*(A)$. Thus for any subset A of r-space,

$$m^*(A) = \inf \{ \sum |I_i| : A \subset \bigcup I_i \} .$$

When A belongs to a certain class of sets to be defined presently, $m^*(A)$ will be called the Lebesgue measure of A, and denoted by $m(A)$.

The edges of the intervals I_i may be closed, open, or half-open, and the sequence of intervals may be finite or infinite. It may happen that the series $\sum |I_i|$ diverges for every sequence $\{I_i\}$ that covers A; in this case $m^*(A) = \infty$. In all other cases $m^*(A)$ is a nonnegative real number.

This definition can be modified in either or both of two respects without affecting the value of $m^*(A)$. In the first place, we may require that the diameters of the intervals I_i should all be less than a given positive number δ. This is clear since each interval can be divided into subintervals of diameter less than δ without affecting the sum of their volumes. Secondly, we may require that all the intervals be open. For any covering sequence $\{I_i\}$ and $\varepsilon > 0$ we can find open intervals J_i such that $I_i \subset J_i$ and $\sum |J_i| \leqq \sum |I_i| + \varepsilon$. Hence the greatest lower bound for open coverings is the same as for all covering sequences.

We shall now deduce a number of properties of outer measure.

Theorem 3.1. *If $A \subset B$ then $m^*(A) \leqq m^*(B)$.*

This is obvious, since any sequence $\{I_i\}$ that covers B also covers A.