NUMERICAL METHODS IN GEOMECHANICS

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edited by

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Institute for Foundation Engineering, Soil Mechanics, Rock Mechanics and Water Ways Construction, Aachen

VOLUME ONE:

1 *Theoretical developments*

2 *Flow and consolidation*

3 *Constitutive laws*

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Preface

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and Water Ways Construction, Aachen, Germany

Within the last few years, a wide range of numerical techniques such as the finite element, finite difference, integral equation and other methods have been developed for solution of complex problems in the various fields of geomechanics.

During the preceding Conferences on Numerical Methods in Geomechanics, the 1st Conference held in Vicksburg, Mississippi (1972) and the 2nd Conference in Blacksburg, Virginia (1976), it became apparent that the numerical methods had developed into an indispensible tool in solving of geomechanical problems.

Consequently the 3rd International Conference in Aachen, Fed. Rep. of Germany, April 2 - 6, 1979, which is organized by the Institute for Foundation Engineering, Soil Mechanics, Rock Mechanics and Water Ways Construction, University (RWTH) Aachen, attracted the interest of numerous experts all over the world. More than 120 contributions on recent research projects and successful applications of numerical methods in soil mechanics, foundation engineering, rock mechanics, geological engineering and related fields of geomechanics have been submitted by experts from 25 countries. These papers are published in the three volumes of the Proceedings, which will be distributed before the conference, thus enabling for the participants a thorough and detailed preparation for the various sessions of the conference.

During the preparatory works it was decided to publish a fourth volume after the conference. This volume will contain interesting discussions and those papers which were not submitted in due time. There will also be an alphabetical list of the participants and contributing authors included in this latter volume.

I would like to thank all authors and those colleagues chairing the various sessions of the meeting for their manifold efforts. I would also like to express my thanks for the considerable support of the members of the "Conference Committee" namely:

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as well as of the "International Committee on Numerical Methods in Geomechanics" namely:

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The assistance of the members of both these Committees was extremely valuable. Finally I like to express my gratitute to the Conference Secretary Mr. Dipl.-Ing. W. Rauscher and to the members of my Institute.

March, 1979

W. Wittke
Chairman

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1 Theoretical developments

Third International Conference on Numerical Methods in Geomechanics / Aachen / 2-6 April 1979

A general procedure for solving three dimensional elasticity problems in geomechanics

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1 INTRODUCTION

Many practical problems in geomechanics and geodynamics, specially those encountered in foundation and earthquake engineering deal with regions containing inhomogeneities delineated within a semi-infinite body. The governing equations for these problems are those of the three dimensional theory of elasticity defined over an unbounded domain. Due to this characteristic, the application of a unique numerical procedure is not possible unless restrictive assumptions are imposed in the analysis. For this, a promising alternative that can handle regions of infinite extent is the simultaneous use of finite elements and boundary elements.

Different ways of connecting finite elements to boundary elements have recently been proposed (Shaw 1977). In all the cases the formulations have been based on the definition of a variational principle, restricting the numerical method of solution to the minimization of a given functional. In this paper the finite element method and a new class of the boundary element methods for three dimensional elasticity are introduced as general solution procedures directly derived from the universally known principle of virtual work. It is shown that since this principle can be viewed as a general statement of the method of weighted residuals for three dimensional elasticity, the true potentiality of the finite and boundary elements may be fully exploited by using the different modalities of the method e.g. the conventional boundary element method is derived by using a nodal collocation approach.

In order to obtain the best results within the limitations of the proposed approximation and to be compatible with existent finite element computer programmes a Galerkin approach for the boundary element method is presented. It is proved that for formulations not requiring integration by parts the application of the method of Galerkin is numerically equivalent to the method of orthogonal collocation, and that in the latter case the number of calculations is drastically reduced.

Boundaries extending to infinity may be modelled by using boundary elements of semi-infinite extent which approximate conditions in the far field. Discontinuities on the boundary stresses such as those found along the edges of foundation mat are handled by using double numeration of nodes plus a continuity condition for the displacements.

The consistency of the entire formulation makes the coupling of finite elements and boundary elements straightforward. In the solution strategy the interface between the region with inhomogeneities and the rest of the earth is considered as a boundary separating an interior region, where finite elements are practicable, from an exterior one whose characteristics are simple enough to admit the use of boundary elements.

An application is devised where the overall problem of interaction arch dam-foundation-reservoir is analized. To the authors' knowledge this problem has not been discussed previously in the literature. A further simplification is foreseen by using asymptotic expansions of the fundamental solutions employed in the approximation. This simplification renders boundary elements for which boundary conditions are approximated locally.

2 NUMERICAL FORMULATION

Consider an elastic body in equilibrium occupying a volume V and limited by the boundary S. The fundamental equations of the theory of three dimensional elasticity which govern its behaviour are

a) Equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + B_i - \rho \ddot{u}_i = 0 \qquad (1)$$

b) Stress-strain relationships

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \qquad (2)$$

c) Strain-displacement relationships

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (3)$$

where

- ε_{ij} strain tensor
- σ_{ij} stress tensor
- B_i body force vector per unit volume
- U strain energy density function expressed in terms of strain components
- u_i displacement vector
- ρ mass density

The boundary conditions associated with this problem may be written 1) for the part of the boundary with prescribed tractions (S_σ), as

$$t_i \equiv \sigma_{ij} n_i = \bar{t}_i \qquad (4)$$

where $\bar{t}_i$ are the prescribed tractions and n_i is a unit vector normal to the surface at the considered point, and, for the part of the boundary with prescribed displacements (S_u) $\bar{u}$

$$u_i = \bar{u}_i \qquad (5)$$

where $\bar{u}_i$ are the prescribed displacements.

An approximate solution for the problem given in eqs 1 to 5 may be efficiently obtained combining a physical principle with an approximate solution method, i.e. the principle of virtual work and Galerkin's method. As shown further on, such analogy allows the simultaneous establishment of the finite element method and the boundary integral method. If one considers that u_i, σ_{ij} and ε_{ij} are independent functions, the principle of virtual work for an elastic body is defined as

$$\iiint_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + B_i - \rho \ddot{u}_i\right) \delta u_i dV + \iiint_V \left(\sigma_{ij} - \frac{\partial U}{\partial \varepsilon_{ij}}\right) \delta \varepsilon_{ij} dV + \iiint_V \left(\varepsilon_{ij} - \frac{\partial u_i}{\partial x_j}\right) \delta \sigma_{ij} dV = \int_{S_\sigma} (\sigma_{ij} n_j - \bar{t}_i) \delta u_i dS - \int_{S_u} (u_i - \bar{u}_i) \delta t_i dS \qquad (6)$$

where δ is the variation of the considered function.

NB. The above equation contains all the fundamental equations previously defined.

Duncan (1938) demonstrates that the principle of virtual work may be used as the fundamental equation for the numerical formulation of static and dynamic problems of deformable bodies. He shows that this principle turns out to be equivalent to the formulation of Galerkin's method with weighting functions obtained from the approximate solution in such a way that the product of the residual times the weighting function has work units. Imposing the condition that the functions u_i, σ_{ij} and ε_{ij} satisfy identically the stress-strain and strain-displacements relationships, the principle of virtual work (Galerkin's method) reduces to

$$\iiint_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + B_i - \rho \ddot{u}_i\right) \delta u_i dV = \int_{S_\sigma} (t_i - \bar{t}_i) \delta u_i dS - \int_{S_u} (u_i - \bar{u}_i) \delta t_i dS \qquad (7)$$

2.1 Finite element method

The difficulty in obtaining approximate solutions for problems with complicated boundary conditions using direct methods of solution has given rise to the extensive use of the finite element method (Zienkiewicz 1977). This consists basically in the discretization of the region under study into a finite number of subregions called finite elements. The overall approximation of the problem is attained using functions localised within each element and which satisfy continuity requirements in the interior boundaries. For a linear elastic medium it may be shown that after integration by parts of the volume integral, the principle of virtual work leads to a weak formulation of Galerkin's method,

$$\iiint \left[\lambda \frac{\partial u_k}{\partial x_k} \frac{\partial \delta u_i}{\partial x_j} + G\left(\frac{\partial u_i}{\partial x_j} \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_i}\right) + \rho \ddot{u}_i \delta u_i\right] dV = \int_{S_\sigma} \bar{t}_i \delta u_i dS + \iiint_V F_i \delta u_i dV - \int_{S_u} (u_i - u_i) \delta t_i dS \qquad (8)$$

For a finite element the displacement

vector may be approximated as

$$u_i = \phi_{i\ell} U_\ell \qquad (9)$$

where $\phi_{i\ell} = \phi_{i\ell}(x_k)$ are interpolation functions satisfying displacement boundary conditions, and U_ℓ the degrees of freedom representing the nodal values of the displacement vector which are the unknowns of the formulation.

Substituting the approximate solution given by eq 9 in eq 8 and taking into account that the variations of the displacement vector are given by the interpolation functions used in the definition of eq 9, it is found that

$$\iiint \left[\lambda \frac{\partial \phi_{k\ell}}{\partial x_k} \frac{\partial \phi_{im}}{\partial x_i} + G \left(\frac{\partial \phi_{i\ell}}{\partial x_k} \frac{\partial \phi_{im}}{\partial x_k} + \frac{\partial \phi_{i\ell}}{\partial x_k} \frac{\partial \phi_{km}}{\partial x_i} \right) \right] dV \, U_\ell + \iiint \rho \phi_{i\ell} \phi_{im} \, dV \, \ddot{U}_\ell = \qquad (10)$$

$$= \int_{S_\sigma} t_i \phi_{im} dS + \iiint F_i \phi_{im} \, dV$$

which may be concisely expressed as

$$k_{\ell m} U_\ell + m_{\ell m} \ddot{U}_\ell = p_m \qquad (11)$$

where $k_{\ell m}$ is the stiffness matrix, $m_{\ell m}$ the mass matrix , and p_m the load vector of the element

If the previous formulation is applied each finite element in the region and an assemblage of their contributions is performed, the following system of equations is obtained

$$K\underset{\sim}{U} + M\ddot{\underset{\sim}{U}} = \underset{\sim}{P} \qquad (12)$$

where K is the global stiffness matrix, M is the global mass matrix, and $\underset{\sim}{P}$ the global load vector of the entire region.

2.2 Boundary integral method

As in the finite element method, the boundary integral method (also referred to in the literature as the Trefftz method or the boundary element method) may be formulated in a general way from the principle of virtual work.

If the functions u_i and σ_{ij} satisfy, a priori, the equilibrium equations in the interior of a- elastic body but do not satisfy the boundary conditions (Trefftz 1926), the principle of virtual work given by eq 7 reads as

$$\int_{S_\sigma} (t_i - \bar{t}_i) \delta u_i - \int_{S_u} (u_i - \bar{u}_i) \delta t_i = 0 \qquad (13)$$

From a numerical point of view, eq 13 is equivalent to the application of Galerkin's method, where only the boundary conditions need be enforeced as the functions which approximate the solution u_i satisfy the differential equations of equilibrium. As in the case of finite elements, the global approximation of the problem is difficult and hence it is convenient to divide the boundary S into subregions and apply eq 13 to each one.

The approximation of the functions in eq 13 may be carried out using the superposition of elemental solutions or Green functions distributed over an auxiliary surface which may or may not coincide with the boundary of the body. The knowledge of elemental solutions which, as well as satisfying the equilibrium equations of the system, also satisfy some boundary conditions (e.g. Mindlin's solution for a concentrated load within an elastic halfspace) generally simplify the application of eq 13. Unfortunately,the scarcity of these solutions and the difficulties involved in obtaining them generally leads to the use of elemental solutions for static or dynamic concentrated loads within an unbounded elastic body

Using the auxiliary surface (S_A) which does not coincide with the boundary, the displacement vector may be defined as the single layer potential (Cruse 1977)

$$u_i(x_k) = \int_{S_A} G_{ij}(x_k, y_k) \, d_j(y_k) dS \qquad (14)$$

where

G_{ij} i component of the displacement function at the point x_k due to a unit force in j direction applied at point y_k

d_j load density function in direction j defined on the auxiliary surface

y_k point on the auxiliary force

The stress tensor associated to G_{ij} may be written from Hooke's law as

$$T_{ijm} = \lambda \frac{G_{\ell j}}{\partial x_\ell} \delta_{im} + G \left(\frac{G_{ij}}{\partial x_m} + \frac{\partial G_{mj}}{\partial x_i} \right) \qquad (15)$$

Hence, the stress tensor σ_{ij} associated with u_i may be defined as the double layer potential

$$\sigma_{im} = \int_{S_A} T_{ijm}(x_k, y_k)\ d_j(y_k)\ dS \qquad (16)$$

In order to evaluate eqs 14 and 16 it is convenient to divide the auxiliary surface S_A into subregions using the same number of nodal points that for the real boundary and for each of these elements the load density d_j may be approximated as

$$d_j = \phi_{j\ell} D_\ell \qquad (17)$$

where $\phi_{j\ell}$ are interpolation functions defined over the considered boundary element and D_ℓ the nodal values of the load density.

Substituting eqs 14, 16 and 17 in eq 13, an approximation is found for the principle of virtual work given by

$$\int_{S_\sigma} \Sigma \int_{S_A} \left[T_{ijm}\ (x_k, y_k)\ n_m \phi_{j\ell} dS D_\ell - \bar{t}_i \right] \delta u_i dS -$$

$$- \int_{S_u} \Sigma \int_{S_A} \left[G_{ij}(x_k, y_k)\ \phi_{j\ell} dS D_\ell - \bar{u}_i \right]$$

$$\delta t_i dS = 0 \qquad (18)$$

The prescribed functions $\bar{t}_i$ and $\bar{u}_i$ may be represented in an approximate way using interpolation functions $\bar{\phi}_{i\ell}$ defined within each element on the boundary S. So that for an element, we have

$$\bar{t}_i = \bar{\phi}_{in} T_n \qquad (19)$$

$$\bar{u}_i = \bar{\phi}_{in} U_n$$

where T_n and U_n represent the prescribed nodal values of the functions $\bar{t}_i$ and $\bar{u}_i$, respectively. For a given boundary element, eq 18 may be written

$$\int_{S_\sigma} \left[\Sigma \int_{S_A} T_{ijm}(x_k, y_k)\ n_m \phi_{j\ell} dS\ D_\ell - \right.$$

$$\left. - \bar{\phi}_{in} T_n \right] \delta u_i dS \qquad (20)$$

$$- \int_{S_u} \left[\Sigma \int_{S_A} G_{ij}\ (x_k, y_k)\ \phi_{j\ell} dS\ D_\ell + \bar{\phi}_{in} U_n \right]$$

$$\delta t_i dS = 0$$

NB. If eq 20 is considered as a general expression for the weighted residuals method, the method of nodal collocation requires the use of weighting functions defined as

$$\delta t_i = \delta_i (x_k - x_k^\alpha)$$

$$\delta u_i = \delta_i (x_k - x_k^\alpha) \qquad (21)$$

where δ_i is the Dirac delta and x_k^α the coordinates of the nodal point α of the element (collocation points).

The inclusion of eq 21 in eq 20 gives rise to a formulation equivalent to the indirect boundary element method, where the boundary and the solution are defined in different surfaces, thus

$$\Sigma \int_{S_A} T_{ijm}\ (x_k^\alpha, y_k)\ n_m \phi_{j\ell} dS\ D_\ell -$$

$$- T_i(x_k^\alpha) = 0 \qquad (22)$$

$$\Sigma \int_{S_A} G_{ij}\ (x_k^\alpha, y_k)\ \phi_{j\ell} dS\ D_\ell - U_i(x_k^\alpha) = 0 \qquad (23)$$

From the previous paragraphs it may be concluded that the numerical formulation given by eq 20 is general, as the selection of different weighting functions leads to diverse versions of the weighted residual method.

In the case of Galerkin's method, the functions δu_i and δt_i on the boundary S are given by

$$\delta u_i = \bar{\phi}_{in} \qquad (24)$$

$$\delta t_i = \bar{\phi}_{in} \qquad (25)$$

Substituting eqs 24 and 25 in eq 20 we get

$$\int_{S_\sigma} \Sigma \int_{S_A} T_{ijm}\ (x_k, y_k)\ n_m \phi_{j\ell} dS_A D_\ell \phi_{ip} dS -$$

$$- \int_{S_\sigma} \phi_{in} \phi_{ip} T_n dS - \int_{S_u} \Sigma \int G_{ij}\ (x_k, y_k)$$

$$\phi_{j\ell} dS_A D_\ell \phi_{ip} dS + \int_{S_u} \phi_{in} \phi_{ip} U_n dS = 0 \qquad (26)$$

or, in compact form.

$$a_{p\ell}D_\ell - \bar{c}_{pn}T_n - b_{p\ell}D_\ell + \bar{c}_{pn}U_n = 0 \quad (27)$$

$$\ell = 1,2,\ldots,N$$

$$p,n = 1,2,\ldots,M$$

where N is the total number of nodal values of the load density applied on the auxiliary surface S_A and M the number of degrees of freedom in the considered boundary element.

Eq 26 allows a boundary element to have stress and displacement restrictions simultaneously. However, if only displacement boundary conditions exist, we have

$$- b_{p\ell}D_\ell + \bar{c}_{pn}U_n = 0 \quad (28)$$

and if only stress boundary conditions, we have

$$a_{p\ell}D_\ell - \bar{c}_{pn}T_n = 0 \quad (29)$$

The assemblage of the corresponding contributions of each boundary element gives a matrix equation similar to that resulting from the application of the finite element method, which is

$$A_{ij}D_j - B_{ij}D_j = \bar{C}_{ij}T_j - \bar{C}_{ij}U_j \quad (30)$$

$$i,j = 1,2,\ldots,N$$

where the matrices A_{ij}, B_{ij} and C_{ij} are the overall matrices resulting from the assemblage of eq 27 for each element. Thus, in order to solve the problem set out in previous paragraphs the solution of an asymmetric system of linear equations is required. Once this equation is solved the formulation to obtain forces and displacements at any point within the body is derived directly from the discrete version of eqs 14 and 16

$$u_i(x_k) = \Sigma \int_{S_A} G_{ij}(x_k,y_k)\phi_{j\ell}D_\ell dS \quad (31)$$

$$\sigma_{ij}(x_k) = \Sigma \int_{S_A} T_{ijm}(x_k,y_k)\phi_{j\ell}D_\ell dS \quad (32)$$

An alternative for the approximate solution of eq 13 is to consider that the auxiliary surface on which the potentials in eqs 14 and 16 are defined coincides with the boundary of the body. In this case it is necessary to consider the discontinuity of the double layer potential when the auxiliary surface approaches the boundary S. This problem has been tackled by Cruse (1977) who found that the single layer potential does not change, i. e.

$$u_i(x_k) = \int_S G_{ij}(x_k,y_k)d_j(y_k)dS \quad (33)$$

However, the double layer potential has a discontinuity on the boundary S measured by

$$\left[\sigma^+_{im}(x_k) - \sigma_{im}(x_k)\right] n_m = 2\beta b_j(x_k) \quad (34)$$

So that eq 16 becomes

$$\sigma_{im}(x_k)n_m = \beta d_j(x_k) + \int_S T_{ijm}(x_k,y_x)n_m d_j(x_k)dS \quad (35)$$

where

$$\beta = \begin{cases} \frac{1}{2} \text{ if } y_k \to x_k & \text{from inside the body} \\ -\frac{1}{2} \text{ if } y_k \to x_k & \text{from outside the body} \end{cases}$$

As in the case in which the auxiliary surface did not coincide with the boundary S, eqs 33 and 35 provide an exact solution for u_i and σ_{im}. The errors which may exist are due to the approximation of the function $d_j(x_k)$ as a stepwise function

3. TREATMENT OF INFINITE BOUNDARIES

Quite a common situation in the numerical formulation of geomechanic problems is the existence of regions limited by boundaries which extend to such distances that makes impractical, and on occasions impossible, the direct application of discrete methods of the boundary element type.

This difficulty, which also appears in applications of the finite element method may be overcome by using boundary elements specially designed to model conditions at infinity.

In this paper it is proposed to use boundary elements of the type used by Ayala (1973), for applications of the finite element method to wave propagation problems. A typical geometry for a plane boundary element, here called semi-infinite, is shown in fig 1. This type of element is connected to the conventional boundary elements along the side $x_1 = 0$.

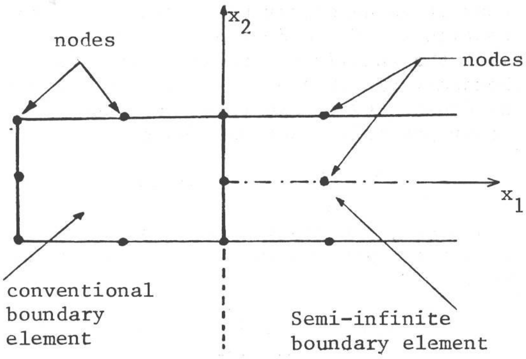

FIGURE 1

In the same way as finite elements, the interpolation functions may be defined by Lagrange polynomials. However, in order to guarantee that the integrals of eq 13 tend to a finite value, it is necessary to multiply the interpolation functions by functions with asymptotic behaviour. For example, the interpolation functions for a boundary element such as that in fig 1, may be defined as a combination of Lagrange polynomials in the direction x_2 and interpolation functions defined along the axis x_1 as

$$N_i = e^{(x_1^i - x_1)/L} \prod_{\substack{j=1 \\ i \neq j}}^{n-i} \frac{(x_1^j - x_1)}{(x_1^j - x_1^i)} \qquad (36)$$

$$i = 1,2,\dots,N-1$$

where L is a parameter which measures the rate at which the exponential function decays.

The interpolation function is obtained from

$$N_N = 1 - \sum_{j=1}^{N-1} N_j \qquad (37)$$

For elements of distorted form it is possible to define a parametric representation similar to that employed in iso-parametric finite and boundary elements.

4. EQUIVALENCE OF GALERKIN'S METHOD AND THE METHOD OF ORTHOGONAL COLLOCATION

The analytical evaluation of the integrals in eq 26 is not practical because of the complexity of the kernels. An alternative way of carrying out these integrations is to use the ideas of numerical integration commonly employed in the finite element method.

The numerical integration of eq 26 is obtained by replacing the integrals over S and S_A by their numerical quadratures. If only the numerical integration over S is shown, we have

$$\Sigma \int_{S_A} T_{ijm}(x_k^\alpha; y_k)\, n_m \phi_{j\ell}(y_k)\, dS\, D_\ell \phi_{ip}(x_k^\alpha) W^\alpha$$

$$- \phi_{in}(x_k^\alpha)\phi_{ip}(x_k^\alpha) W^\alpha T_n -$$

$$- \Sigma \int G_{ij}(x_k^\alpha, y_k)\, \phi_{j\ell}(y_k)\, dS\, D_\ell \phi_{ip}(x_k^\alpha) W^\alpha +$$

$$\phi_{in}(x_k^\alpha)\phi_{ip}(x_k^\alpha) W^\alpha U_n = 0 \qquad (38)$$

where the superscript α indicates the considered integration point and W^α the corresponding weighting factor.

For simplicity, eq 38 can be written as

$$E_{i\ell} D_\ell \phi_{ip}(x_k^\alpha) W^\alpha - \phi_{in}(x_k^\alpha) T_n \phi_{ip}(x_k^\alpha) W^\alpha -$$

$$\qquad (39)$$

$$- F_{i\ell} D_\ell \phi_{ip}(x_k^\alpha) W^\alpha + \phi_{in}(x_k^\alpha) U_n \phi_{ip}(x_k^\alpha) W^\alpha = 0$$

where

$$E_{i\ell} = \Sigma \int_{S_A} T_{ijm}(x_k^\alpha, y_k)\, n_m \phi_{j\ell}(y_k)\, dS \qquad (40)$$

$$F_{i\ell} = \Sigma \int_{S_A} G_{ij}(x_k^\alpha, y_k)\, \phi_{j\ell}(y_k)\, dS \qquad (41)$$

Notice that the number of operations involved in eq 39 is excessive. This disadvantage, however, may be overcome if instead of Galerkin's method another version of the method of weighted residuals -known as orthogonal collocation-is used. In this method the weighting functions are Dirac deltas evaluated at points corresponding to the zeros of the Legendre polynomials, which for boundary elements coincide with the points of Gaussian integration, x_k . For semi-infinite elements the points of collocation are the zeros of the Laguerre polynomials.

The application of the method of orthogonal collocation in eq 20 leads to