Proceedings of the Symposium on

ALGEBRAIC GEOMETRY IN East Asia

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Akira Ohbuchi
Tokushima University, Japan

Kazuhiro Konno
Sampei Usui
Osaka University, Japan

Atsushi Moriwaki
Kyoto University, Japan

Noboru Nakayama
RIMS, Japan

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ALGEBRAIC GEOMETRY IN East Asia

Preface

This book is the proceedings of the conference "Algebraic Geometry in East Asia" which was held in International Institute for Advanced Studies (IIAS) (9-3 Kizugawadai, Kizu-cho, Soraku-gun, Kyoto 619-0225, Japan), during August 3 to August 10, 2001. The conference was partially supported by Grants-in-Aid for Scientific Researches ((A) (1) 11304001 by Sampei Usui and (B) (2) 13440008 by Atsushi Moriwaki).

Although many east Asian mathematicians now play a leading role in the international mathematical community, in modern times mathematics did not become a strength of east Asian scholarship until the early 20'th century. In particular, as a consequence of this relatively short history, the various east Asian mathematical communities, such as the algebraic geometers represented at this conference, have had less of a chance to meet and exchange ideas as their western counterparts. Accordingly, one of the primary goals of the conference was to facilitate such an exchange.

As the breadth of the topics covered in this proceedings demonstrate, the conference was indeed successful in assembling a wide spectrum of east Asian mathematicians, and gave them a welcome chance to discuss current state of algebraic geometry. It is the first time that such a conference has been held in algebraic geometry, and we hope that it is but the start of continuing tradition.

We wish to thank, first of all, the lecturers for their beautiful talks. We also wish to thank the participants for their cooperation and providing stimulating atmosphere.

During the conference, administrative staffs in IIAS as well as many graduate students from Osaka University and Kyoto University helped us. Without them, the conference would have been less successful than we had hoped for.

In particular, we wish to thank Ms. Yoshiko Kusaki and Ms. Minako Tanaka of the IIAS as well as the following graduate students from Osaka and Kyoto Universities: Mr. Masao Aoki, Mr. Takeshi Harui, Mr. Atsushi Ikeda, Mr. Michiaki Inaba, Mr. Tomokazu Kawahara, Mr. Hiraku Kawanoe, Mr. Shinya Kitagawa, Mr. Masaaki Murakami, Mr. Hiroto Nakayama, Ms. Noriko Tsuda and Mr. Daisuke Yanase.

Finally, we would like to thank Prof. Junjiro Kanamori, the Director of IIAS, for his dignified opening speech.

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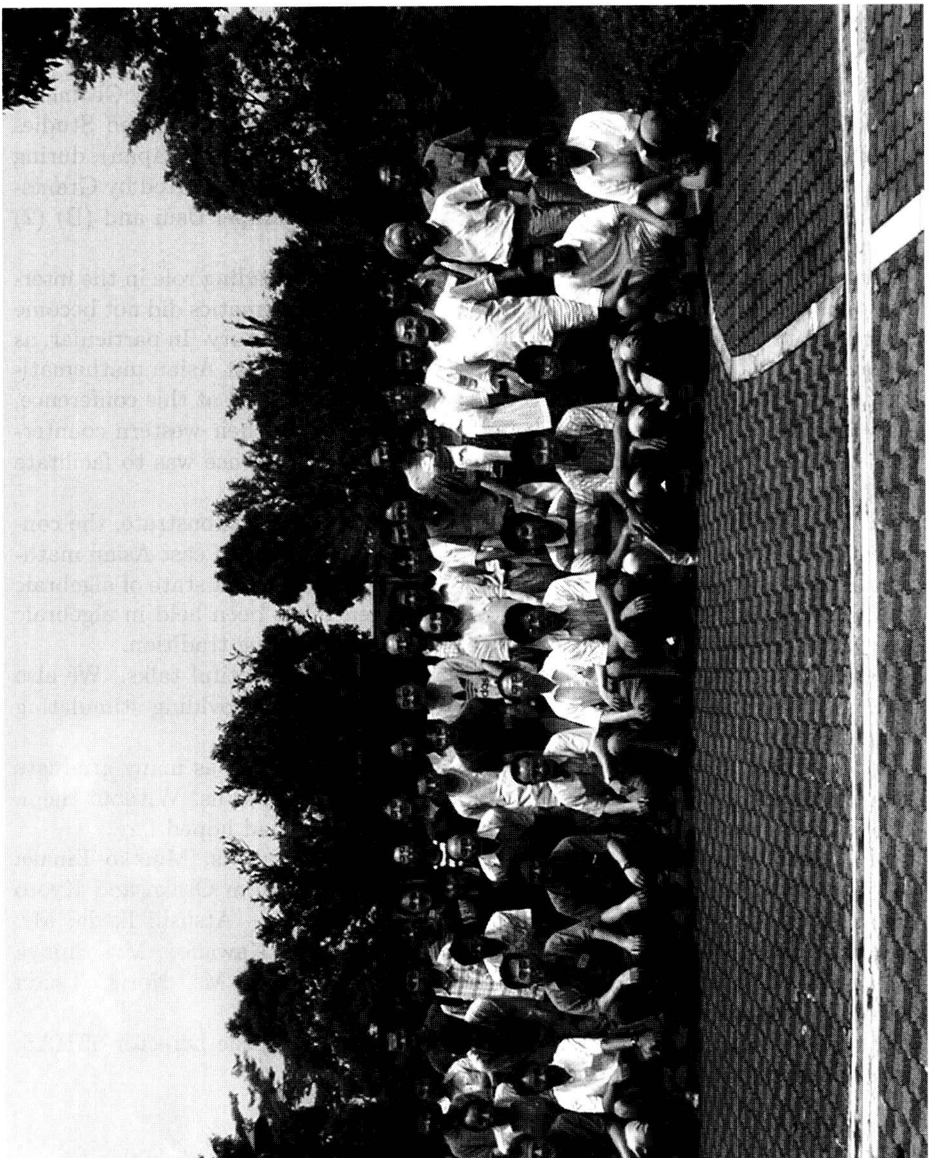

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SHU KAWAGUCHI, ATSUSHI MORIWAKI AND KAZUHIKO YAMAKI

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INTRODUCTION

This note was first written in Japanese for intensive lectures of Arakelov geometry organized by Moriwaki from December 8 to December 10, 1998 at Kyoto University. These lectures were intended to give an overview of Arakelov geometry and a proof of Bogomolov's conjecture for general algebraic geometers. From that time, we have considered that this note should be available for not only Japanese but also a broader range of readers. We, however, had no chance to translate it into English. Fortunately, during the meeting "Algebraic Geometry in East Asia," Professor Usui recommended its translation. Here we would like to express hearty thanks for his proposal.

The final goal of this note is a generalization of conjectures of Bogomolov and Lang. For this purpose, in the first part, we introduce "Arithmetic Chow group" and "Arithmetic Riemann-Roch theorem," in which we do not give a rigorous proof for every result, but we believe that this is a good introduction of Arakelov geometry. In the middle part, we consider "Existence of a small section," "Adelic metric and admissible pairing" and "Arithmetic height function," in which several techniques of Arakelov geometry are used. In the final part, we give a proof of Bogomolov's conjecture and a generalization of conjectures of Bogomolov and Lang.

Here we would like to explain a generalization of conjectures of Bogomolov and Lang in the case of a curve and its Jacobian. Let K be a number field, X a geometrically irreducible projective curve of genus greater than or equal to 2 over K, and J the Jacobian of X. Let us fix an embedding $\iota : X(\overline{K}) \to J(\overline{K})$ and a Néron-Tate pairing

$$\langle\ ,\ \rangle : J(\overline{K}) \times J(\overline{K}) \to \mathbb{R}.$$

Let Γ be a subgroup of $J(\overline{K})$ with $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} < \infty$. Let $(\Gamma \otimes \mathbb{R})^{\perp}$ be the orthogonal complement of $\Gamma \otimes \mathbb{R}$ in $J(\overline{K}) \otimes \mathbb{R}$ in terms of the Néron-Tate pairing. Let ι_Γ : $X(\overline{K}) \to (\Gamma \otimes \mathbb{R})^{\perp}$ be the compositions of maps

$$X(\overline{K}) \xrightarrow{\ \iota\ } J(\overline{K}) \longrightarrow J(\overline{K}) \otimes \mathbb{R} \xrightarrow{\text{projection}} (\Gamma \otimes \mathbb{R})^{\perp}.$$

Then, a generalization of conjectures of Bogomolov and Lang says that the fiber of ι_Γ is finite and the image of ι_Γ is a discrete subset of $(\Gamma \otimes \mathbb{R})^{\perp}$ in terms of the metric arising from the Néron-Tate pairing. If we consider the case $\Gamma = J(K)$, then the first assertion is nothing more than Mordell's conjecture. Moreover, if we consider the case $\Gamma = \{0\}$, then we have Bogomolov's conjecture.

§1, §2, §3 and §7 were written by Kawaguchi, §4 by Yamaki, and §5 and §6 by Moriwaki. We hope that this note will be useful for anyone who wants to know Arakelov geometry.

1. ARITHMETIC CHOW GROUP

1.1. **Introduction.** In the Arakelov geometry, one considers, roughly speaking,

- schemes over $\mathbb{Z}$ with "infinity," instead of algebraic varieties over a field,
- vector bundles with "metrics at infinity," instead of vector bundles.

Szpiro [34] wrote about the Arakelov geometry that

> Put metrics at infinity on vector bundles and you will have a geometric intuition of compact varieties to help you.

Let us see this analogy by comparing a compact Riemann surface with $\mathrm{Spec}(\mathbb{Z})$.

Let X be a compact Riemann surface.

(A) Let f be a nonzero rational function on X. Then we have

$$\mathrm{div}(f) = \sum_{p \in X} v_p(f) \cdot [p] \in \bigoplus_{p \in X} \mathbb{Z} \cdot [p] =: \mathrm{Div}(X).$$

It follows from the residue formula that

$$\deg(f) = \sum_{p \in X} v_p(f) = 0.$$

Set $\mathrm{Rat}(X) = \{\mathrm{div}(f) \mid f \in \mathbb{C}(X)^*\} \subset \mathrm{Div}(X)$, and $\mathrm{CH}^1(X) = \mathrm{Div}(X)/\mathrm{Rat}(X)$. Then the map $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ induces the map $\deg : \mathrm{CH}^1(X) \to \mathbb{Z}$.

(B) Let L be a holomorphic line bundle over X, and s a nonzero rational section of L. Put

$$\mathrm{div}(s) = \sum_{p \in X} v_p(s) \cdot [p] \in \mathrm{CH}^1(X).$$

Then, as an element of $\mathrm{CH}^1(X)$, $\mathrm{div}(s)$ depends only on L, i.e., it is independent of the choice of s.

(C) Let $\mathrm{Pic}(X)$ be the set of isomorphism classes of holomorphic line bundles over X. Then, we have the isomorphism

$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X), \quad L \mapsto \mathrm{div}(s),$$

where s is any nonzero rational section of L. In particular, through this isomorphism, $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ is defined.

Next, we consider $\mathcal{X} = \mathrm{Spec}(\mathbb{Z})$.

(A') Since we consider a scheme at "infinity," let us set

$$\widehat{\mathrm{Div}}(\mathcal{X}) = \left(\bigoplus_{p:\text{prime}} \mathbb{Z} \cdot [p] \right) \oplus \mathbb{R} \cdot [\infty].$$

Let $f \in \mathbb{Q}$ be a nonzero rational number. We set

$$v_\infty(f) = -\log |f|^2 \in \mathbb{R},$$

$$\widehat{\mathrm{div}}(f) = \sum_{p:\text{prime}} v_p(f) \cdot [p] + v_\infty(f) \cdot [\infty] \in \widehat{\mathrm{Div}}(\mathcal{X}).$$

We define $\widehat{\deg} : \widehat{\mathrm{Div}}(\mathcal{X}) \to \mathbb{R}$ by

$$\sum_{p:\text{prime}} n_p \cdot [p] + a \cdot [\infty] \mapsto \sum_{p:\text{prime}} n_p \log p + \frac{1}{2}a.$$

It follows from the product formula that

$$\widehat{\deg}(\widehat{\mathrm{div}}(f)) = \sum_{p:\text{prime}} v_p(f) \log p + \frac{1}{2}v_\infty(f) = 0.$$

(In this case, the product formula is an obvious consequence of the prime factorization.) Set

$$\widehat{\mathrm{Rat}}(\mathcal{X}) = \{\widehat{\mathrm{div}}(f) \mid f \in \mathbb{Q} \setminus \{0\}\} \subset \widehat{\mathrm{Div}}(\mathcal{X})$$

and $\widehat{\mathrm{CH}}^1(\mathcal{X}) = \widehat{\mathrm{Div}}(\mathcal{X}) / \widehat{\mathrm{Rat}}(\mathcal{X})$. Then the map $\widehat{\deg} : \widehat{\mathrm{Div}}(\mathcal{X}) \to \mathbb{Z}$ induces the map $\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathcal{X}) \to \mathbb{Z}$.

(B') Let $\mathcal{L}$ be a line bundle over X. Since we consider a "metric at infinity," let us take a hermitian metric

$$h : \mathcal{L}_{\mathbb{C}} \times \mathcal{L}_{\mathbb{C}} \longrightarrow \mathbb{C}$$

on $\mathcal{L}_{\mathbb{C}} = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{C}$. We denote the pair $(\mathcal{L}, h)$ by $\overline{\mathcal{L}}$, and call it a hermitian line bundle.

Let s be a nonzero rational section of $\mathcal{L}$. Put

$$\widehat{\mathrm{div}}(s) = \sum_{p:\text{prime}} v_p(s) \cdot [p] + (-\log h(s_{\mathbb{C}}, s_{\mathbb{C}})) \cdot [\infty] \in \widehat{\mathrm{CH}}^1(X).$$

Then, as an element of $\widehat{\mathrm{CH}}^1(\mathcal{X})$, $\widehat{\mathrm{div}}(s)$ depends only on $\mathcal{L}$, i.e., it is independent of the choice of s.

(C') Two hermitian line bundles $\overline{\mathcal{L}}_1 = (\mathcal{L}_1, h_1)$ and $\overline{\mathcal{L}}_2 = (\mathcal{L}_2, h_2)$ are said to be isomorphic if there exist an isomorphism $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ of line bundles such that the induced map $\phi_{\mathbb{C}} : (\mathcal{L}_{1\mathbb{C}}, h_1) \to (\mathcal{L}_{2\mathbb{C}}, h_2)$ is an isometry.

Let $\widehat{\mathrm{Pic}}(\mathcal{X})$ be the set of isomorphism classes of hermitian line bundles over $\mathcal{X}$. Then

$$\widehat{c}_1 : \widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{CH}}^1(\mathcal{X}), \quad \overline{\mathcal{L}} \mapsto \widehat{\mathrm{div}}(s)$$

is an isomorphism, where s is any nonzero rational section of $\mathcal{L}$ (cf. Proposition 1.3.4). In particular, through this isomorphism, $\widehat{\deg} : \widehat{\mathrm{Pic}}(\mathcal{X}) \to \mathbb{Z}$ is defined.

To sum up, by adding "infinity" to $\mathrm{Spec}(\mathbb{Z})$ and considering hermitian line bundles over $\mathrm{Spec}(\mathbb{Z})$, one has the degree map $\widehat{\deg}$ for $\mathrm{Spec}(\mathbb{Z})$, similar to the degree map deg for a compact Riemann surface, in the sense that $\widehat{\deg}(\widehat{\mathrm{div}}(f)) = 0$ for $f \in \mathbb{Q} \setminus \{0\}$.

Such an analogy between the ring of integers of a number field and a compact Riemann surface have been noted by many mathematicians such as Hasse and Weil to name a few. Arakelov [2] generalized this analogy to a 2-dimensional case, and established the intersection theory on arithmetic surfaces, which corresponds to that on projective surfaces over $\mathbb{C}$. Then, Faltings proved in [5] among other things a Riemann-Roch theorem on arithmetic surfaces. Gillet and Soulé (cf. [9] [11], [3], [12], [13]) developed its higher dimensional theory, including arithmetic cycles and their intersections on arithmetic varieties, arithmetic Chern classes of hermitian vector bundles, and an arithmetic Riemann-Roch theorem. Such theory of arithmetic varieties is called the Arakelov geometry. We remark that [32] is a good reference of the Arakelov geometry and [33] is a good quick guide to it.

Let us consider a case of dimension ≥ 2. We set

$$\mathcal{X} = \operatorname{Proj}\left(\mathbb{Z}[X,Y,Z]/(Y^2Z = X^3 + XZ^2)\right).$$

This is an example of arithmetic surfaces (cf. §1.3). To consider "infinity" means to consider the compact Riemann surface

$$\mathcal{X}_{\mathbb{C}} = \operatorname{Proj}\left(\mathbb{C}[X,Y,Z]/(Y^2Z - X^3 - XZ^2)\right).$$

Moreover, to consider a line bundle with a "metric at infinity" means to consider a pair $\overline{\mathcal{L}} = (\mathcal{L}, h)$, where $\mathcal{L}$ is a line bundle over $\mathcal{X}$ and h is a hermitian metric on $\mathcal{L}_{\mathbb{C}}$.

Then, what is $\widehat{\operatorname{Div}}(\mathcal{X})$? The case $\mathcal{X} = \operatorname{Spec}(\mathbb{Z})$ kept in mind, it would be natural (undoubtedly with hindsight) to think that

$$(\operatorname{div}(s), -\log h(s_{\mathbb{C}}, s_{\mathbb{C}}))$$

becomes an "arithmetic divisor" on $\mathcal{X}$, where s is a nonzero section of $\mathcal{L}$.

In what follows, we will give the precise definitions of arithmetic varieties, arithmetic divisors, the arithmetic Chow groups on an arithmetic variety etc., due to Gillet and Soulé. In fact, an arithmetic divisor on an arithmetic variety $\mathcal{X}$ is a pair $(\mathcal{Z}, g)$ such that $\mathcal{Z}$ is a cycle on $\mathcal{X}$ and g is a "Green current" on $\mathcal{X}(\mathbb{C})$; And $(\operatorname{div}(s), -\log h(s_{\mathbb{C}}, s_{\mathbb{C}}))$ above is indeed an arithmetic divisor on $\mathcal{X}$. So, let us first define Green currents in the next subsection.

1.2. **Currents.** Let X be a d-dimensional compact complex manifold. Let $A^{p,q}(X)$ be the space of C^∞ differential forms of type (p,q) on X. We endow $A^{p,q}(X)$ with the compact C^∞ topology: Namely, a sequence $\{\eta_n\}$ converges to η_∞ in $A^{p,q}(X)$ if and only if (1) there exists a compact set K such that for any n the support of η_n is contained in K and (2) any order derivation of η_n uniformly converges to the corresponding derivation of η_∞.

Definition 1.2.1. We call a continuous linear functional

$$T : A^{d-p,d-q}(X) \to \mathbb{C}$$

a *current* of type (p,q) on X. Let $D^{p,q}(X)$ be the set of currents of type (p,q) on X.

Example 1.2.2. For $\omega \in A^{p,q}(X)$, set

$$[\omega] : A^{d-p,d-q}(X) \to \mathbb{C}, \quad \eta \mapsto \int_X \omega \wedge \eta.$$

Then, $[\omega] \in D^{p,q}(X)$. Via $[\cdot] : A^{p,q}(X) \hookrightarrow D^{p,q}(X), \omega \mapsto [\omega]$, $A^{p,q}(X)$ is regarded as a subspace of $D^{p,q}(X)$.

Example 1.2.3. Let ω be a differential form of type (p,q) on X with locally integrable coefficients. Then, in the same way as in Example 1.2.2, one obtains the current $[\omega] \in D^{p,q}(X)$.

Example 1.2.4. Let X be a non-singular projective variety over $\mathbb{C}$ and Y a subvariety of X of codimension p. Then, we have the Dirac type current $\delta_Y \in D^{p,p}(X)$ defined by

$$\delta_Y : A^{d-p,d-p}(X) \to \mathbb{C}, \quad \eta \mapsto \int_{Y^{ns}} \eta,$$

where Y^{ns} is the set of non-singular points of Y. Note that, if $\pi : \widetilde{Y} \to Y$ is a resolution of singularities of Y, then the equality $\delta_Y(\eta) = \int_{\widetilde{Y}} \pi^*(\eta)$ holds, and thus $\int_{Y^{ns}} \eta$ converges.

Example 1.2.5. More generally, let $Y = \sum_\alpha n_\alpha Y_\alpha$ ($n_\alpha \in \mathbb{Z}$) be a cycle of codimension p on X. Then, we have the current $\delta_Y \in D^{p,p}(X)$ defined by $\delta_Y = \sum_\alpha n_\alpha \delta_{Y_\alpha}$.

A current $T \in D^{p,p}(X)$ is said to be *real* if $T(\overline{\eta}) = \overline{T(\eta)}$ for any $\eta \in A^{p,p}(X)$. For example, δ_Y as above is a real current.

Let us define differential operators on $\bigoplus_{p,q} D^{p,q}(X)$. For $T \in D^{p,q}(X)$, we define $\partial T \in D^{p+1,q}(X)$ and $\overline{\partial} T \in D^{p,q+1}(X)$ by

$$\partial T(\eta) = (-1)^{p+q+1} T(\partial \eta) \qquad (\eta \in A^{d-(p+1),d-q}(X)),$$
$$\overline{\partial} T(\eta) = (-1)^{p+q+1} T(\overline{\partial} \eta) \qquad (\eta \in A^{d-p,d-(q+1)}(X)).$$

We see from the Stokes theorem that $[\partial\omega] = \partial[\omega]$ and $[\overline{\partial}\omega] = \overline{\partial}[\omega]$ for $\omega \in A^{p,q}(X)$. Moreover, we set

$$d = \partial + \overline{\partial},$$
$$d^c = \frac{1}{4\pi\sqrt{-1}}(\partial - \overline{\partial}).$$

Note that $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$. For $\omega \in A^{p,q}(X)$, we similarly have $[d\omega] = d[\omega]$ and $[d^c\omega] = d^c[\omega]$.

The pull-back of differential forms induces the push-forward of currents. Indeed, let $\pi : X \to Y$ be a holomorphic map of compact complex manifolds and

g an element of $D^{p,q}(X)$. Then, the push-forward of g, which is an element of $D^{p+\dim X-\dim Y, q+\dim X-\dim Y}$ and is denoted by $\pi_*(g)$, is defined by $\pi_*(g)(\eta) = g(\pi^*\eta)$.

Definition 1.2.6 (Green Current). Let X be a compact Kähler manifold and $Z \subset X$ a cycle of codimension p. A *Green current* for Z is a current $g \in D^{p-1,p-1}(X)$ such that there exists $\omega \in A^{p,p}(X)$ with

$$dd^c(g) + \delta_Z = [\omega].$$

Let X be a compact Kähler manifold and L a line bundle over X. A C^∞-hermitian metric h on L is a C^∞-field of hermitian inner products in the fibers of L. Namely, for each $x \in X$,

$$h_x : L_x \times L_x \longrightarrow \mathbb{C}$$

is a hermitian inner product, and h_x is C^∞ with respect to x. We call $\overline{L} := (L, h)$ a *C^∞-hermitian line bundle* .

Example 1.2.7. Let X be a smooth projective variety over $\mathbb{C}$, $\overline{L} = (L, h)$ a C^∞-hermitian line bundle over X, and s a nonzero rational section of L. Then, since $-\log h(s,s)$ is locally integrable, $[-\log h(s,s)]$ defines a current in $D^{0,0}(X)$ (cf. Example 1.2.3). The following Poincaré-Lelong formula shows that $[-\log h(s,s)]$ is actually a Green current for $\mathrm{div}(s)$.

Theorem 1.2.8 (Poincaré-Lelong formula). *Let X be a smooth projective variety over $\mathbb{C}$, $\overline{L} = (L, h)$ a C^∞-hermitian line bundle over X, and s a nonzero rational section of L. Let $c_1(\overline{L}) \in A^{1,1}(X)$ be the first Chern form of $\overline{L}$. Then, the following formula holds in $D^{1,1}(X)$:*

(1.2.8.1) $$dd^c[-\log h(s,s)] + \delta_{\mathrm{div}(s)} = [c_1(\overline{L})].$$

Proof: Let d be the dimension of X.

Step 1 The assertion holds if the support of $\mathrm{div}(s)$ is a normal crossing divisor. Indeed, for any $p \in X$, one can take an open neighborhood U of p and local coordinates $z_1, \dots, z_d$ of U such that $\mathrm{Supp}(\mathrm{div}(s))$ is locally defined by $z_1 z_2 \cdots z_k = 0$. By the partition of unity and the linearity, it suffices to show that, for any $\eta \in A^{d-1,d-1}(U)$ with compact support,

$$\int_U \log|z_1|^2 \, dd^c\eta = \int_{z_1=0} \eta.$$

We will show this equality in the appendix (cf. Lemma A.1).

Step 2 We treat a general case. Set $D = \mathrm{div}(s)$. By Hironaka's theorem [17], there exists a proper morphism $\pi : \widetilde{X} \to X$ such that

(i) $\widetilde{X}$ is smooth,
(ii) $E = \pi^*(D)_{red}$ is a normal crossing divisor,

(iii) $\pi|_{\widetilde{X}\setminus \mathrm{Supp}(E)} : \widetilde{X} \setminus \mathrm{Supp}(E) \to X \setminus \mathrm{Supp}(D)$ is isomorphic.

On the other hand, we have

$$\int_X -\log h(s,s)\, dd^c\eta = \int_{\widetilde{X}} -\log \pi^* h(\pi^* s, \pi^* s)\, dd^c(\pi^*\eta)$$

$$\int_X c_1(\overline{L}) \wedge \eta = \int_{\widetilde{X}} c_1(\pi^*\overline{L}) \wedge \pi^*\eta.$$

We write $E = \overline{D} + E'$, where $\overline{D}$ is the strict transform of D. Then $\dim \pi_*(E') < \dim D$. Thus, we have

$$\int_{\mathrm{div}(\pi^* s)} \pi^*\eta = \int_{\overline{D}} \pi^*\eta + \int_{E'} \pi^*\eta = \int_D \eta = \int_{\mathrm{div}(s)} \eta.$$

Since $\mathrm{div}(\pi^* s)_{red}$ is a normal crossing divisor, by Step 1, we obtain the formula in a general case. □

Remark 1.2.9. In relation to the last part of the proof of Theorem 1.2.8, we remark that, for any morphism $\pi : X \to Y$ of compact complex manifolds and cycle Z of X, we have $\pi_*(\delta_Z) = \delta_{\pi_*(Z)}$.

In the rest of this subsection, we consider some basic properties of Green currents.

Lemma 1.2.10 (dd^c-lemma for currents). *Let X be a compact Kähler manifold and η an element of $D^{p,p}(X)$. Assume that η is d-exact. Then, there exists $\gamma \in D^{p-1,p-1}(X)$ such that $\eta = dd^c\gamma$.*

For its proof, we refer to [14, p149], where the dd^c-lemma for C^∞ differential forms is proven. Since operators $\partial, \overline{\partial}^*, G_{\overline{\partial}}$ for C^∞ differential forms in [14, p149] are all extended to those for currents, the same argument goes for the dd^c-lemma for currents.

Proposition 1.2.11. *Let X be a compact Kähler manifold. Then, for any cycle Z of codimension p on X, there exists a Green current for Z.*

Proof: Take $\omega \in A^{p,p}(X)$ which represents Z in the cohomology class. Then, $[\omega] - \delta_Z$ is d-exact. By the dd^c-lemma, there exists a current $g \in D^{p-1,p-1}(X)$ with $[\omega] - \delta_Z = dd^c g$. □

Proposition 1.2.12. *Let X be a compact Kähler manifold and Z a cycle of codimension p on X. Let g_1 and g_2 be Green currents for Z. Then, there exist $\eta \in A^{p,p}(X)$, $u \in D^{p-2,p-1}(X)$ and $v \in D^{p-1,p-2}(X)$ such that*

$$g_1 - g_2 = [\eta] + \partial u + \overline{\partial} v.$$

Proof: For $i = 1, 2$, write $dd^c(g_i) + \delta_Y = [\omega_i]$ for some $\omega_i \in A^{p,p}(X)$. Then, we have $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(g_1 - g_2) = [\omega_1 - \omega_2]$. Then, the assertion follows from the following lemma. □

Lemma 1.2.13. *Let X be a compact Kähler manifold and g an element of $D^{p,q}(X)$. Assume that $\partial\overline{\partial}g$ is C^∞, i.e., $\partial\overline{\partial}g = [\omega]$ for some $\omega \in A^{p+1,q+1}(X)$. Then, there exist $\eta \in A^{p,q}(X)$, $u \in D^{p-1,q}(X)$ and $v \in D^{p,q-1}(X)$ such that*

$$g = [\eta] + \partial u + \overline{\partial} v.$$

Proof: By [14, p385], $d, \partial, \overline{\partial}$-cohomology of currents coincide with d, ∂, $\overline{\partial}$-cohomology of C^∞ differential forms.

Thus, if $\partial\overline{\partial}g = [\omega]$ for some $\omega \in A^{p+1,q+1}(X)$, then there exists a C^∞ differential form α with $\omega = \partial\alpha$. Since $\partial(\overline{\partial}g - [\alpha]) = 0$, there exist a C^∞ differential form β and a current g_1 such that $\overline{\partial}g - [\alpha] = [\beta] + \partial g_1$. Thus, $\partial\overline{\partial}g_1 = \overline{\partial}[\alpha+\beta] = [\overline{\partial}(\alpha+\beta)]$, where g_1 is a current of type $(p-1, q+1)$. By iterating this procedure, we get a current g_n of type $(p-n, q+n)$ and a C^∞ differential form α_n that satisfy $\overline{\partial}g_n = [\alpha_n] + \partial g_{n+1}$ $(n \geq 1)$.

Since $g_{n+1} = 0$ for $n \geq p$, we have $\overline{\partial}g_n = [\alpha_n]$. Since α_n is a C^∞ differential form, there exists a C^∞ differential form η_n with $g_n = [\eta_n] + \overline{\partial}v_n$. Then, since $\overline{\partial}(g_{n-1} + \partial v_n)$ $= [\alpha_{n-1}] + \partial g_n - \partial(g_n - [\eta_n]) = [\alpha_{n-1}] + \partial[\eta_n]$, there exists a C^∞ differential form η_{n-1} with $g_{n-1} = [\eta_{n-1}] + \partial u_{n-1} + \overline{\partial}v_{n-1}$. By iterating this procedure, we get $g = [\eta] + \partial u + \overline{\partial}v$ for some C^∞ differential form η. □

In Example 1.2.7, a C^∞-hermitian line bundle $\overline{L} = (L, h)$ and a nonzero rational section s of L determine a Green current for the divisor $\mathrm{div}(s)$. The next proposition shows that the converse also holds.

Proposition 1.2.14. *Let X be a smooth projective variety over $\mathbb{C}$ and D a divisor on X. Let s be a rational section of $\mathcal{O}_X(D)$ with $\mathrm{div}(s) = D$. Let g be a Green current for D. Then, there exist a C^∞-hermitian metric h over $\mathcal{O}_X(D)$ with*

$$g = [-\log h(s, s)].$$

Proof: Take any C^∞-hermitian metric h' on $\mathcal{O}_X(D)$. By Example 1.2.7, $[-\log h'(s, s)]$ is a Green current for D. Since D is a divisor, by Proposition 1.2.12, there exists a C^∞ function f with

$$g - [-\log h'(s, s)] = [f].$$

Set $h = \exp(-f)h'$. Then, h is a desired C^∞-hermitian metric over $\mathcal{O}_X(D)$. □