

Earl D. Rainville

Phillip E. Bedient

Elementary Differential Equations

FIFTH EDITION

Elementary Differential Equations

FIFTH EDITION

Earl D. Rainville

Late Professor of Mathematics
University of Michigan

Phillip E. Bedient

Professor of Mathematics
Franklin and Marshall College

Copyright © 1974, Macmillan Publishing Co., Inc.
Printed in the United States of America

All rights reserved. No part of this book may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, recording, or any information storage and retrieval system, without permission in writing from the Publisher.

Earlier editions copyright 1949 and 1952, © 1958, and copyright © 1964 and 1969 by Macmillan Publishing Co., Inc.
Some material is from *The Laplace Transform: An Introduction*, copyright © 1963 by Earl D. Rainville.

Macmillan Publishing Co., Inc.
866 Third Avenue, New York, New York 10022
Collier-Macmillan Canada, Ltd., Toronto, Ontario

Library of Congress Cataloging in Publication Data

Rainville, Earl David.
Elementary differential equations.

"Chapters 1 through 16 of this book [also] appear separately as A short course in differential equations, 5th ed."

I. Differential equations. I. Bedient, Phillip Edward, 1922-- joint author. II. Title.

QA371.R29 1974
ISBN 0-02-397720-5

515'.35

72-12749

Printing:

8

Year:

8 9

Preface

to the Fifth Edition

The presentation of a new edition of Professor Rainville's book gives testimony to the effect of the simple and direct style that has made earlier editions most popular with students of differential equations. The present author has attempted to maintain that simplicity of style while making several modest changes and one major addition. The chapter on systems of equations has been greatly expanded to include the use of matrix techniques in solving systems of linear equations with constant coefficients.

As in all of the previous editions, an attempt has been made to maintain a balance between developing techniques for solving equations and the theory necessary to support those techniques. Both technique and theory are illustrated in numerous applications interspersed throughout the book.

The material is arranged to permit great flexibility in the choice of topics for a semester course. Except for Chapters 1, 2, 5, 16 through 18, and either 6 and 7 or 10 and 11, any chapter on ordinary differential equations can be omitted without interfering with the study of later chapters. Parts of chapters can be omitted in many instances.

For a course that aims at reaching power series as rapidly as is consistent with some treatment of more elementary methods a reasonable syllabus should include Chapters 1 and 2, Chapters 5, 6, 7, 8, parts of Chapters 13 and 15, Chapters 17 and 18, and whatever applications the instructor cares to insert.

This book has sufficient material for a full year course, if the individual topics are taken up with the attention to detail that such a course suggests.

Chapters 1 through 16 of this book appear separately as *A Short Course in Differential Equations*, Fifth Edition. The shorter version is intended for courses that do not include discussion of infinite series methods.

The author wishes to express his appreciation for the many suggestions made by colleagues at Franklin and Marshall College and by students and instructors at other colleges and universities. He is pleased to acknowledge in particular the thoughtful assistance he received from Professor Richard Howland of Rhode Island College.

P. E. B.

Lancaster, Pennsylvania

Contents

1 Definitions, Elimination of Arbitrary Constants

1. Examples of differential equations 1
2. Definitions 3
3. The elimination of arbitrary constants 5
4. Families of curves 9

2 Equations of Order One

5. The isoclines of an equation 16
6. An existence theorem 19
7. Separation of variables 20
8. Homogeneous functions 25
9. Equations with homogeneous coefficients 27
10. Exact equations 30
11. The linear equation of order one 35

- 12. The general solution of a linear equation 39
- Miscellaneous exercises 41

3 Elementary Applications

- 13. Velocity of escape from the earth 45
- 14. Newton's law of cooling 47
- 15. Simple chemical conversion 48
- 16. Orthogonal trajectories; rectangular coordinates 53
- 17. Orthogonal trajectories; polar coordinates 56

4 Additional Topics on Equations of Order One

- 18. Integrating factors found by inspection 59
- 19. The determination of integrating factors 63
- 20. Substitution suggested by the equation 68
- 21. Bernoulli's equation 70
- 22. Coefficients linear in the two variables 73
- 23. Solutions involving nonelementary integrals 78
- Miscellaneous exercises 80

5 Linear Differential Equations

- 24. The general linear equation 82
- 25. Linear independence 83
- 26. An existence and uniqueness theorem 84
- 27. The Wronskian 84
- 28. General solution of a homogeneous equation 87
- 29. General solution of a nonhomogeneous equation 89
- 30. Differential operators 90
- 31. The fundamental laws of operation 93
- 32. Some properties of differential operators 94

6 Linear Equations with Constant Coefficients

- 33. Introduction 98
- 34. The auxiliary equation; distinct roots 98
- 35. The auxiliary equation; repeated roots 101
- 36. A definition of $\exp z$ for imaginary z 105

- 37. The auxiliary equation; imaginary roots 106
- Miscellaneous exercises 109

7 Nonhomogeneous Equations: Undetermined Coefficients

- 38. Construction of a homogeneous equation from a specified solution 111
- 39. Solution of a nonhomogeneous equation 114
- 40. The method of undetermined coefficients 116
- 41. Solution by inspection 122

8 Variation of Parameters

- 42. Introduction 127
- 43. Reduction of order 128
- 44. Variation of parameters 132
- 45. Solution of $y'' + y = f(x)$ 136
- Miscellaneous exercises 139

9 Inverse Differential Operators

- 46. The exponential shift 140
- 47. The operator $1/f(D)$ 144
- 48. Evaluation of $[1/f(D)]e^{ax}$ 145
- 49. Evaluation of $(D^2 + a^2)^{-1} \sin ax$ and $(D^2 + a^2)^{-1} \cos ax$ 146

10 The Laplace Transform

- 50. The transform concept 150
- 51. Definition of the Laplace transform 151
- 52. Transforms of elementary functions 152
- 53. Transforming initial value problems 156
- 54. Sectionally continuous functions 159
- 55. Functions of exponential order 161
- 56. Functions of class A 164
- 57. Transforms of derivatives 166
- 58. Derivatives of transforms 169
- 59. The gamma function 170
- 60. Periodic functions 171

11 Inverse Transforms

- 61. Definition of an inverse transform 177
- 62. A step function 181
- 63. A convolution theorem 186
- 64. Simple initial value problems 190
- 65. Special integral equations 197

12 Applications

- 66. Vibration of a spring 204
- 67. Undamped vibrations 207
- 68. Resonance 211
- 69. Damped vibrations 213
- 70. The simple pendulum and the deflection of beams 218

13 Systems of Equations

- 71. Introduction 225
- 72. Elementary elimination calculus 225
- 73. First order systems with constant coefficients 229
- 74. Solution of a first order system 231
- 75. Some matrix algebra 232
- 76. First order systems revisited 239
- 77. Complex eigenvalues 246
- 78. Repeated eigenvalues 251
- 79. Nonhomogeneous systems 257
- 80. The Laplace transform 260

14 Electric Circuits and Networks

- 81. Circuits 266
- 82. Simple networks 269

15 The Existence and Uniqueness of Solutions

- 83. Preliminary remarks 279
- 84. An existence and uniqueness theorem 280
- 85. A Lipschitz condition 282

- 86. A proof of the existence theorem 283
- 87. A proof of the uniqueness theorem 286
- 88. Other existence theorems 288

16 Nonlinear Equations

- 89. Preliminary remarks 289
- 90. Factoring the left member 290
- 91. Singular solutions 293
- 92. The c -discriminant equation 294
- 93. The p -discriminant equation 296
- 94. Eliminating the dependent variable 298
- 95. Clairaut's equation 300
- 96. Dependent variable missing 303
- 97. Independent variable missing 305
- 98. The catenary 308
- Miscellaneous exercises 310

17 Power Series Solutions

- 99. Linear equations and power series 312
- 100. Convergence of power series 314
- 101. Ordinary points and singular points 316
- 102. Validity of the solutions near an ordinary point 317
- 103. Solutions near an ordinary point 318

18 Solutions Near Regular Singular Points

- 104. Regular singular points 329
- 105. The indicial equation 332
- 106. Form and validity of the solutions near a regular singular point 334
- 107. Indicial equation with difference of roots nonintegral 334
- 108. Differentiation of a product of functions 340
- 109. Indicial equation with equal roots 341
- 110. Indicial equation with difference of roots a positive integer, nonlogarithmic case 348
- 111. Indicial equation with difference of roots a positive integer, logarithmic case 352
- 112. Solution for large x 357
- 113. Many-term recurrence relations 362

- 114. Summary 366
 - Miscellaneous exercises 368
- 19 Equations of Hypergeometric Type
 - 115. Equations to be treated in this chapter 370
 - 116. The factorial function 370
 - 117. The hypergeometric equation 371
 - 118. Laguerre polynomials 373
 - 119. Bessel's equation with index not an integer 374
 - 120. Bessel's equation with index an integer 375
 - 121. Hermite polynomials 377
 - 122. Legendre polynomials 377
 - 123. The confluent hypergeometric equation 379
- 20 Numerical Methods
 - 124. General remarks 381
 - 125. The increment method 382
 - 126. A method of successive approximation 384
 - 127. An improvement on the preceding method 386
 - 128. The use of Taylor's theorem 387
 - 129. The Runge-Kutta method 389
 - 130. A continuing method 392
- 21 Partial Differential Equations
 - 131. Remarks on partial differential equations 396
 - 132. Some partial differential equations of applied mathematics 397
 - 133. Method of separation of variables 399
 - 134. A problem on the conduction of heat in a slab 404
- 22 Orthogonal Sets
 - 135. Orthogonality 410
 - 136. Simple sets of polynomials 411
 - 137. Orthogonal polynomials 412
 - 138. Zeros of orthogonal polynomials 413
 - 139. Orthogonality of Legendre polynomials 414
 - 140. Other orthogonal sets 416

23 Fourier Series

- 141. Orthogonality of a set of sines and cosines 418
- 142. Fourier series: an expansion theorem 421
- 143. Numerical examples of Fourier series 425
- 144. Fourier sine series 434
- 145. Fourier cosine series 438
- 146. Numerical Fourier analysis 442
- 147. Improvement in rapidity of convergence 443

24 Boundary Value Problems

- 148. The one-dimensional heat equation 445
- 149. Experimental verification of the validity of the heat equation 451
- 150. Surface temperature varying with time 454
- 151. Heat conduction in a sphere 456
- 152. The simple wave equation 458
- 153. Laplace's equation in two dimensions 461

25 Additional Properties of the Laplace Transform

- 154. Power series and inverse transforms 465
- 155. The error function 469
- 156. Bessel functions 476
- 157. Differential equations with variable coefficients 479

26 Partial Differential Equations; Transform Methods

- 158. Boundary value problems 480
- 159. The wave equation 485
- 160. Diffusion in a semi-infinite solid 487
- 161. Canonical variables 490
- 162. Diffusion in a slab of finite width 492
- 163. Diffusion in a quarter-infinite solid 496

1

Definitions, Elimination of Arbitrary Constants

1. Examples of differential equations

In physics, engineering, and chemistry and, on occasion, in such subjects as biology, physiology, and economics it is necessary to build a mathematical model to represent certain problems. It is often the case that these mathematical models involve the search for an unknown function that satisfies an equation in which derivatives of the unknown function play an important role. Such equations are called differential equations. As in equation (3) below, a derivative may be involved implicitly through the presence of differentials. Our aim is to find methods for solving differential equations; that is, to find the unknown function or functions that satisfy the differential equation.

The following are examples of differential equations:

$$\frac{dy}{dx} = \cos x, \quad (1)$$

$$\frac{d^2y}{dx^2} + k^2y = 0, \quad (2)$$

$$(x^2 + y^2) dx - 2xy dy = 0, \quad (3)$$

$$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (4)$$

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = E \omega \cos \omega t, \quad (5)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad (6)$$

$$\left(\frac{d^2 w}{dx^2} \right)^3 - xy \frac{dw}{dx} + w = 0, \quad (7)$$

$$\frac{d^3 x}{dy^3} + x \frac{dx}{dy} - 4xy = 0, \quad (8)$$

$$\frac{d^2 y}{dx^2} + 7 \left(\frac{dy}{dx} \right)^3 - 8y = 0, \quad (9)$$

$$\frac{d^2 y}{dt^2} + \frac{d^2 x}{dt^2} = x, \quad (10)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf. \quad (11)$$

When an equation involves one or more derivatives with respect to a particular variable, that variable is called an *independent* variable. A variable is called *dependent* if a derivative of that variable occurs.

In the equation

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = E \omega \cos \omega t \quad (5)$$

i is the dependent variable, t the independent variable, and L , R , C , E , and ω are called parameters. The equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (6)$$

has one dependent variable V and two independent variables.

Since the equation

$$(x^2 + y^2) dx - 2xy dy = 0 \quad (3)$$

may be written

$$x^2 + y^2 - 2xy \frac{dy}{dx} = 0$$

or

$$(x^2 + y^2) \frac{dx}{dy} - 2xy = 0,$$

we may consider either variable to be dependent, the other being the independent one.

Oral Exercise

Identify the independent variables, the dependent variables, and the parameters in the equations given as examples in this section.

2. Definitions

The *order* of a differential equation is the order of the highest-ordered derivative appearing in the equation. For instance,

$$\frac{d^2y}{dx^2} + 2b \left(\frac{dy}{dx} \right)^3 + y = 0 \quad (1)$$

is an equation of "order two." It is also referred to as a "second-order equation."

More generally, the equation

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad (2)$$

is called an "*n*th-order" ordinary differential equation. Under suitable restrictions on the function F , equation (2) can be solved explicitly for $y^{(n)}$ in terms of the other $n + 1$ variables $x, y, y', \dots, y^{(n-1)}$, to obtain

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}). \quad (3)$$

For the purposes of this book we shall assume that this is always possible. Otherwise, an equation of the form of equation (2) may actually represent more than one equation of the form of equation (3).

For example, the equation

$$x(y')^2 + 4y' - 6x^2 = 0$$

actually represents the two different equations,

$$y' = \frac{-2 + \sqrt{4 + 6x^3}}{x} \quad \text{or} \quad y' = \frac{-2 - \sqrt{4 + 6x^3}}{x}.$$

A function ϕ , defined on an interval $a < x < b$, is called a solution of the differential equation (3), providing the n derivatives of the function exist on

the interval $a < x < b$ and

$$\phi^{(n)}(x) = f(x, \phi(x), \phi'(x), \dots, \phi^{(n-1)}(x)),$$

for every x in $a < x < b$.

For example, let us verify that

$$y = e^{2x}$$

is a solution of the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0. \quad (4)$$

We substitute our tentative solution into the left member of equation (4) and find that

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 4e^{2x} + 2e^{2x} - 6e^{2x} \equiv 0,$$

which completes the desired verification.

All of the equations we shall consider in Chapter 2 are of order one, and hence may be written

$$\frac{dy}{dx} = f(x, y).$$

For such equations it is sometimes convenient to use the definitions of elementary calculus to write the equation in the form

$$M(x, y) dx + N(x, y) dy = 0. \quad (5)$$

A very important concept is that of the linearity or nonlinearity of a differential equation. An equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is called linear if the function F is a linear function of the variables $y, y', \dots, y^{(n)}$. Thus, the general linear equation of order n may be written

$$b_0(x) \frac{d^n y}{dx^n} + b_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + b_{n-1}(x) \frac{dy}{dx} + b_n(x)y = R(x). \quad (6)$$

For example, equation (1) above is nonlinear, and equation (4) is linear. The equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 4x^3$$

is also linear. The manner in which the independent variable enters the equation has nothing to do with the property of linearity.

Oral Exercises

For each of the following, state whether the equation is ordinary or partial, linear or nonlinear, and give its order.

1. $\frac{d^2x}{dt^2} + k^2x = 0.$
2. $\frac{\partial^2w}{\partial t^2} = a^2 \frac{\partial^2w}{\partial x^2}.$
3. $(x^2 + y^2)dx + 2xydy = 0.$
4. $y' + P(x)y = Q(x).$
5. $y''' - 3y' + 2y = 0.$
6. $yy'' = x.$
7. $\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} + \frac{\partial^2u}{\partial z^2} = 0.$
8. $\frac{d^4y}{dx^4} = w(x).$
9. $x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} = c_1.$
10. $L \frac{di}{dt} + Ri = E.$
11. $(x + y)dx + (3x^2 - 1)dy = 0.$
12. $x(y'')^3 + (y')^4 - y = 0.$
13. $\left(\frac{d^3w}{dx^3}\right)^2 - 2\left(\frac{dw}{dx}\right)^4 + yw = 0.$
14. $\frac{dy}{dx} = 1 - xy + y^2.$
15. $y'' + 2y' - 8y = x^2 + \cos x.$
16. $a da + b db = 0.$

3. The elimination of arbitrary constants

In practice, differential equations arise in many ways, some of which we shall encounter later. There is one way of arriving at a differential equation, however, that is useful in that it gives us a feeling for the kinds of solutions to be expected. In this section we shall start with a relation involving arbitrary constants and, by elimination of those arbitrary constants, come to a differential equation consistent with the original relation. In a sense we start with the answer and find the problem.

Methods for the elimination of arbitrary constants vary with the way in which the constants enter the given relation. A method that is efficient for one problem may be poor for another. One fact persists throughout. Because each differentiation yields a new relation, the number of derivatives that need be used is the same as the number of arbitrary constants to be eliminated. We shall in each case determine the differential equation that is

- (a) Of order equal to the number of arbitrary constants in the given relation.
- (b) Consistent with that relation.
- (c) Free from arbitrary constants.