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# Elementary Differential Equations

FIFTH EDITION

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# Preface

## to the Fifth Edition

The presentation of a new edition of Professor Rainville's book gives testimony to the effect of the simple and direct style that has made earlier editions most popular with students of differential equations. The present author has attempted to maintain that simplicity of style while making several modest changes and one major addition. The chapter on systems of equations has been greatly expanded to include the use of matrix techniques in solving systems of linear equations with constant coefficients.

As in all of the previous editions, an attempt has been made to maintain a balance between developing techniques for solving equations and the theory necessary to support those techniques. Both technique and theory are illustrated in numerous applications interspersed throughout the book.

The material is arranged to permit great flexibility in the choice of topics for a semester course. Except for Chapters 1, 2, 5, 16 through 18, and either 6 and 7 or 10 and 11, any chapter on ordinary differential equations can be omitted without interfering with the study of later chapters. Parts of chapters can be omitted in many instances.

For a course that aims at reaching power series as rapidly as is consistent with some treatment of more elementary methods a reasonable syllabus should include Chapters 1 and 2, Chapters 5, 6, 7, 8, parts of Chapters 13 and 15, Chapters 17 and 18, and whatever applications the instructor cares to insert.

This book has sufficient material for a full year course, if the individual topics are taken up with the attention to detail that such a course suggests.

Chapters 1 through 16 of this book appear separately as *A Short Course in Differential Equations*, Fifth Edition. The shorter version is intended for courses that do not include discussion of infinite series methods.

The author wishes to express his appreciation for the many suggestions made by colleagues at Franklin and Marshall College and by students and instructors at other colleges and universities. He is pleased to acknowledge in particular the thoughtful assistance he received from Professor Richard Howland of Rhode Island College.

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# 1

# Definitions, Elimination of Arbitrary Constants

## 1. Examples of differential equations

In physics, engineering, and chemistry and, on occasion, in such subjects as biology, physiology, and economics it is necessary to build a mathematical model to represent certain problems. It is often the case that these mathematical models involve the search for an unknown function that satisfies an equation in which derivatives of the unknown function play an important role. Such equations are called differential equations. As in equation (3) below, a derivative may be involved implicitly through the presence of differentials. Our aim is to find methods for solving differential equations; that is, to find the unknown function or functions that satisfy the differential equation.

The following are examples of differential equations:

$$\frac{dy}{dx} = \cos x, \tag{1}$$

$$\frac{d^2y}{dx^2} + k^2y = 0, \tag{2}$$

$$(x^2 + y^2)\,dx - 2xy\,dy = 0, \tag{3}$$

$$\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4}$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\cos\omega t, \tag{5}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{6}$$

$$\left(\frac{d^2 w}{dx^2}\right)^3 - xy\frac{dw}{dx} + w = 0, \tag{7}$$

$$\frac{d^3 x}{dy^3} + x\frac{dx}{dy} - 4xy = 0, \tag{8}$$

$$\frac{d^2 y}{dx^2} + 7\left(\frac{dy}{dx}\right)^3 - 8y = 0, \tag{9}$$

$$\frac{d^2 y}{dt^2} + \frac{d^2 x}{dt^2} = x, \tag{10}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{11}$$

When an equation involves one or more derivatives with respect to a particular variable, that variable is called an *independent* variable. A variable is called *dependent* if a derivative of that variable occurs.

In the equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E\omega\cos\omega t \tag{5}$$

$i$ is the dependent variable, $t$ the independent variable, and $L$, $R$, $C$, $E$, and $\omega$ are called parameters. The equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{6}$$

has one dependent variable $V$ and two independent variables.

Since the equation

$$(x^2 + y^2)\,dx - 2xy\,dy = 0 \tag{3}$$

may be written

$$x^2 + y^2 - 2xy\frac{dy}{dx} = 0$$

or

$$(x^2 + y^2)\frac{dx}{dy} - 2xy = 0,$$

we may consider either variable to be dependent, the other being the independent one.

### *Oral Exercise*

Identify the independent variables, the dependent variables, and the parameters in the equations given as examples in this section.

## 2. Definitions

The *order* of a differential equation is the order of the highest-ordered derivative appearing in the equation. For instance,

$$\frac{d^2y}{dx^2} + 2b\left(\frac{dy}{dx}\right)^3 + y = 0 \tag{1}$$

is an equation of "order two." It is also referred to as a "second-order equation."

More generally, the equation

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{2}$$

is called an "$n$th-order" ordinary differential equation. Under suitable restrictions on the function $F$, equation (2) can be solved explicitly for $y^{(n)}$ in terms of the other $n + 1$ variables $x, y, y', \ldots, y^{(n-1)}$, to obtain

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{3}$$

For the purposes of this book we shall assume that this is always possible. Otherwise, an equation of the form of equation (2) may actually represent more than one equation of the form of equation (3).

For example, the equation

$$x(y')^2 + 4y' - 6x^2 = 0$$

actually represents the two different equations,

$$y' = \frac{-2 + \sqrt{4 + 6x^3}}{x} \quad \text{or} \quad y' = \frac{-2 - \sqrt{4 + 6x^3}}{x}.$$

A function $\phi$, defined on an interval $a < x < b$, is called a solution of the differential equation (3), providing the $n$ derivatives of the function exist on the interval $a < x < b$ and

$$\phi^{(n)}(x) = f(x, \phi(x), \phi'(x), \ldots, \phi^{(n-1)}(x)),$$

for every $x$ in $a < x < b$.

For example, let us verify that

$$y = e^{2x}$$

is a solution of the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0. \tag{4}$$

We substitute our tentative solution into the left member of equation (4) and find that

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 4\,e^{2x} + 2\,e^{2x} - 6\,e^{2x} \equiv 0,$$

which completes the desired verification.

All of the equations we shall consider in Chapter 2 are of order one, and hence may be written

$$\frac{dy}{dx} = f(x, y).$$

For such equations it is sometimes convenient to use the definitions of elementary calculus to write the equation in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{5}$$

A very important concept is that of the linearity or nonlinearity of a differential equation. An equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is called linear if the function $F$ is a linear function of the variables $y, y', \ldots, y^{(n)}$. Thus, the general linear equation of order $n$ may be written

$$b_0(x)\frac{d^ny}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x). \tag{6}$$

For example, equation (1) above is nonlinear, and equation (4) is linear. The equation

$$x^2y'' + xy' + (x^2 - n^2)y = 4x^3$$

is also linear. The manner in which the independent variable enters the equation has nothing to do with the property of linearity.

### *Oral Exercises*

For each of the following, state whether the equation is ordinary or partial, linear or nonlinear, and give its order.

**1.** $\dfrac{d^2x}{dt^2} + k^2x = 0.$

**2.** $\dfrac{\partial^2 w}{\partial t^2} = a^2\dfrac{\partial^2 w}{\partial x^2}.$

**3.** $(x^2 + y^2)\,dx + 2xy\,dy = 0.$

**4.** $y' + P(x)y = Q(x).$

**5.** $y''' - 3y' + 2y = 0.$

**6.** $yy'' = x.$

**7.** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

**8.** $\dfrac{d^4y}{dx^4} = w(x).$

**9.** $x\dfrac{d^2y}{dt^2} - y\dfrac{d^2x}{dt^2} = c_1.$

**10.** $L\dfrac{di}{dt} + Ri = E.$

**11.** $(x + y)\,dx + (3x^2 - 1)\,dy = 0.$

**12.** $x(y'')^3 + (y')^4 - y = 0.$

**13.** $\left(\dfrac{d^3w}{dx^3}\right)^2 - 2\left(\dfrac{dw}{dx}\right)^4 + yw = 0.$

**14.** $\dfrac{dy}{dx} = 1 - xy + y^2.$

**15.** $y'' + 2y' - 8y = x^2 + \cos x.$

**16.** $a\,da + b\,db = 0.$

## 3. The elimination of arbitrary constants

In practice, differential equations arise in many ways, some of which we shall encounter later. There is one way of arriving at a differential equation, however, that is useful in that it gives us a feeling for the kinds of solutions to be expected. In this section we shall start with a relation involving arbitrary constants and, by elimination of those arbitrary constants, come to a differential equation consistent with the original relation. In a sense we start with the answer and find the problem.

Methods for the elimination of arbitrary constants vary with the way in which the constants enter the given relation. A method that is efficient for one problem may be poor for another. One fact persists throughout. Because each differentiation yields a new relation, the number of derivatives that need be used is the same as the number of arbitrary constants to be eliminated. We shall in each case determine the differential equation that is

(a) Of order equal to the number of arbitrary constants in the given relation.
(b) Consistent with that relation.
(c) Free from arbitrary constants.