TRENDS IN MATHEMATICS

Differential Equations with Symbolic Computation

Dongming Wang
Zhiming Zheng
Editors

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Editors:

Dongming Wang
School of Science
Beihang University
37 Xueyuan Road
Beijing 100083
China

and

Laboratoire d'Informatique de Paris 6
Université Pierre et Marie Curie - CNRS
8, rue due Capitaine Scott
75015 Paris
France

e-mail: Dongming.Wang@lip6.fr

Zhiming Zheng
School of Science
Beihang University
37 Xueyuan Road
Beijing 100083
China

e-mail: zzheng@pku.edu.cn

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TRENDS IN MATHEMATICS

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Preface

This book provides a picture of what can be done in differential equations with advanced methods and software tools of symbolic computation. It focuses on the symbolic-computational aspect of three kinds of fundamental problems in differential equations: transforming the equations, solving the equations, and studying the structure and properties of their solutions. Modern research on these problems using symbolic computation, or more restrictively using computer algebra, has become increasingly active since the early 1980s when effective algorithms for symbolic solution of differential equations were proposed, and so were computer algebra systems successfully applied to perturbation, bifurcation, and other problems. Historically, symbolic integration, the simplest case of solving ordinary differential equations, was already the target of the first computer algebra package SAINT in the early 1960s.

With 20 chapters, the book is structured into three parts with both tutorial surveys and original research contributions: the first part is devoted to the qualitative study of differential systems with symbolic computation, including stability analysis, establishment of center conditions, and bifurcation of limit cycles, which are closely related to Hilbert's sixteenth problem. The second part is concerned with symbolic solutions of ordinary and partial differential equations, for which normal form methods, reduction and factorization techniques, and the computation of conservation laws are introduced and used to aid the search. The last part is concentrated on the transformation of differential equations into such forms that are better suited for further study and application. It includes symbolic elimination and triangular decomposition for systems of ordinary and partial differential polynomials. A 1991 paper by Wen-tsün Wu on the construction of Gröbner bases based on Riquier–Janet's theory, published in China and not widely available to the western readers, is reprinted as the last chapter. This book should reflect the current state of the art of research and development in differential equations with symbolic computation and is worth reading for researchers and students working on this interdisciplinary subject of mathematics and computational science. It may also serve as a reference for everyone interested in differential equations, symbolic computation, and their interaction.

The idea of compiling this volume grew out of the Seminar on Differential Equations with Symbolic Computation (DESC 2004), which was held in Beijing, China in April 2004 (see http://www-calfor.lip6.fr/~wang/DESC2004) to facilitate the interaction between the two disciplines. The seminar brought together active researchers and graduate students from both disciplines to present their work and to report on their new results and findings. It also provided a forum for over 50 participants to exchange ideas and views and to discuss future development and cooperation. Four invited talks were given by Michael Singer, Lan Wen, Wen-tsün Wu, and Zhifen Zhang. The enthusiastic support of the seminar speakers and the high quality of their presentations are some of the primary motivations for our endeavor to prepare a coherent and comprehensive volume with most recent advances on the subject for publication. In addition to the seminar speakers, several distinguished researchers who were invited to attend the seminar but could not make their trip have also contributed to the present book. Their contributions have helped enrich the contents of the book and make the book beyond a proceedings volume. All the papers accepted for publication in the book underwent a formal review-revision process.

DESC 2004 is the second in a series of seminars, organized in China, on various subjects interacted with symbolic computation. The first seminar, held in Hefei from April 24–26, 2002, was focused on geometric computation and a book on the same subject has been published by World Scientific. The third seminar planned for April 2006 will be on symbolic computation in education.

The editors gratefully acknowledge the support provided by the Schools of Science and Advanced Engineering at Beihang University and the Key Laboratory of Mathematics, Informatics and Behavioral Semantics of the Chinese Ministry of Education for DESC 2004 and the preparation of this book. Our sincere thanks go to the authors for their contributions and cooperation, to the referees for their expertise and timely help, and to all colleagues and students who helped for the organization of DESC 2004.

Beijing
May 2005

Dongming Wang
Zhiming Zheng

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Symbolic Computation of Lyapunov Quantities and the Second Part of Hilbert's Sixteenth Problem

Stephen Lynch

Abstract. This tutorial survey presents a method for computing the Lyapunov quantities for Liénard systems of differential equations using symbolic manipulation packages. The theory is given in detail and simple working MATLAB and Maple programs are listed in this chapter. In recent years, the author has been contacted by many researchers requiring more detail on the algorithmic method used to compute focal values and Lyapunov quantities. It is hoped that this article will address the needs of those and other researchers. Research results are also given here.

Mathematics Subject Classification (2000). Primary 34C07; Secondary 37M20.

Keywords. Bifurcation, Liénard equation, limit cycle, Maple, MATLAB, small-amplitude.

1. Introduction

Poincaré began investigating isolated periodic cycles of planar polynomial vector fields in the 1880s. However, the general problem of determining the maximum number and relative configurations of limit cycles in the plane has remained unresolved for over a century. In the engineering literature, limit cycles in the plane can correspond to steady-state behavior for a physical system (see [25], for example), so it is important to know how many possible steady states there are. There are applications in aircraft flight dynamics and surge in jet engines, for example.

In 1900, David Hilbert presented a list of 23 problems to the International Congress of Mathematicians in Paris. Most of the problems have been solved, either completely or partially. However, the second part of the sixteenth problem remains unsolved. Ilyashenko [37] presents a centennial history of Hilbert's 16th problem and Li [19] has recently written a review article.

The Second Part of Hilbert's Sixteenth Problem. Consider planar polynomial systems of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1.1}$$

where P and Q are polynomials in x and y. The question is to estimate the maximal number and relative positions of the limit cycles of system (1.1). Let H_n denote the maximum possible number of limit cycles that system (1.1) can have when P and Q are of degree n. More formally, the Hilbert numbers H_n are given by

$$H_n = \sup\{\pi(P,Q) : \partial P, \partial Q \leq n\},$$

where ∂ denotes "the degree of" and $\pi(P,Q)$ is the number of limit cycles of system (1.1).

Dulac's Theorem states that a given polynomial system cannot have infinitely many limit cycles. This theorem has only recently been proved independently by Ecalle et al. [13] and Ilyashenko [36], respectively. Unfortunately, this does not imply that the Hilbert numbers are finite.

Of the many attempts to make progress in this question, one of the more fruitful approaches has been to create vector fields with as many isolated periodic orbits as possible using both local and global bifurcations [3]. There are relatively few results in the case of general polynomial systems even when considering local bifurcations. Bautin [1] proved that no more than three small-amplitude limit cycles could bifurcate from a critical point for a quadratic system. For a homogeneous cubic system (no quadratic terms), Sibirskii [33] proved that no more than five small-amplitude limit cycles could be bifurcated from one critical point. Recently, Zoladek [39] found an example where 11 limit cycles could be bifurcated from the origin of a cubic system, but he was unable to prove that this was the maximum possible number.

Although easily stated, Hilbert's sixteenth problem remains almost completely unsolved. For quadratic systems, Songling Shi [32] has obtained a lower bound for the Hilbert number $H_2 \geq 4$. A possible global phase portrait is given in Figure 1. The line at infinity is included and the properties on this line are determined using Poincaré compactification, where a polynomial vector field in the plane is transformed into an analytic vector field on the 2-sphere. More detail on Poincaré compactification can be found in [27]. There are three small-amplitude limit cycles around the origin and at least one other surrounding another critical point. Some of the parameters used in this example are very small.

Blows and Rousseau [4] consider the bifurcation at infinity for polynomial vector fields and give examples of cubic systems having the following configurations:

$$\{(4),1\}, \{(3),2\}, \{(2),5\}, \{(4),2\}, \{(1),5\} \text{ and } \{(2),4\},$$

where $\{(l),L\}$ denotes the configuration of a vector field with l small-amplitude limit cycles bifurcated from a point in the plane and L large-amplitude limit cycles

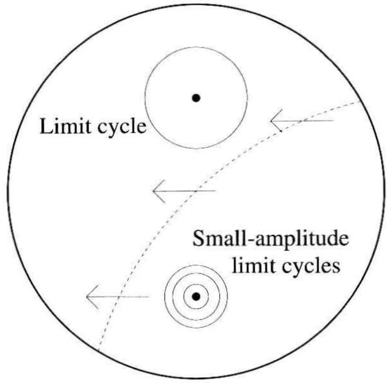

FIGURE 1. A possible configuration for a quadratic system with four limit cycles: one of large amplitude and three of small amplitude.

simultaneously bifurcated from infinity. There are many other configurations possible, some involving other critical points in the finite part of the plane as shown in Figure 2. Recall that a limit cycle must contain at least one critical point.

By considering cubic polynomial vector fields, in 1985, Jibin Li and Chunfu Li [18] produced an example showing that $H_3 \geq 11$ by bifurcating limit cycles out of homoclinic and heteroclinic orbits; see Figure 2.

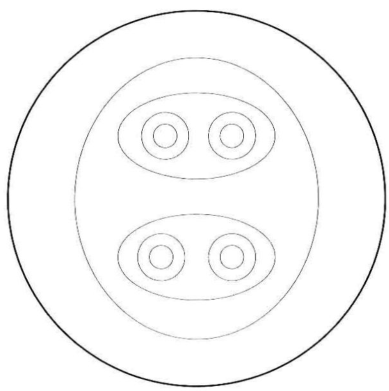

FIGURE 2. A possible configuration for a cubic system with 11 limit cycles.

Returning to the general problem, in 1995, Christopher and Lloyd [7] considered the rate of growth of H_n as n increases. They showed that H_n grows at least as rapidly as $n^2 \log n$.

In recent years, the focus of research in this area has been directed at a small number of classes of systems. Perhaps the most fruitful has been the Liénard system. A method for computing focal values and Lyapunov quantities for Liénard systems is given in detail in the next section. Liénard systems provide a very suitable starting point as they do have ubiquity for systems in the plane [14, 16, 28].

2. Small-Amplitude Limit Cycle Bifurcations

The general problem of determining the maximum number and relative configurations of limit cycles in the plane has remained unresolved for over a century. Both local and global bifurcations have been studied to create vector fields with as many limit cycles as possible. All of these techniques rely heavily on symbolic manipulation packages such as Maple, and MATLAB and its Symbolic Math Toolbox. Unfortunately, the results in the global case number relatively few. Only in recent years have many more results been found by restricting the analysis to *small-amplitude limit cycle* bifurcations.

It is well known that a nondegenerate critical point, say $\mathbf{x_0}$, of center or focus type can be moved to the origin by a linear change of coordinates, to give

$$\dot{x} = \lambda x - y + p(x,y), \quad \dot{y} = x + \lambda y + q(x,y), \tag{2.1}$$

where p and q are at least quadratic in x and y. If $\lambda \neq 0$, then the origin is structurally stable for all perturbations.

Definition 2.1. A critical point, say $\mathbf{x_0}$, is called a *fine focus* of system (1.1) if it is a center for the linearized system at $\mathbf{x_0}$. Equivalently, if $\lambda = 0$ in system (2.1), then the origin is a fine focus.

In the work to follow, *assume that the unperturbed system does not have a center at the origin.* The technique used here is entirely local; limit cycles bifurcate out of a fine focus when its stability is reversed by perturbing λ and the coefficients arising in p and q. These are said to be local or small-amplitude limit cycles. How close the origin is to being a center of the nonlinear system determines the number of limit cycles that may be obtained from bifurcation. The method for bifurcating limit cycles will be given in detail here.

By a classical result, there exists a Lyapunov function, $V(x,y) = V_2(x,y) + V_4(x,y) + \cdots + V_k(x,y) + \cdots$ say, where V_k is a homogeneous polynomial of degree k, such that

$$\frac{dV}{dt} = \eta_2 r^2 + \eta_4 r^4 + \cdots + \eta_{2i} r^{2i} + \cdots, \tag{2.2}$$

where $r^2 = x^2 + y^2$. The η_{2i} are polynomials in the coefficients of p and q and are called the *focal values*. The origin is said to be a fine focus of order k if $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$ but $\eta_{2k+2} \neq 0$. Take an analytic transversal through the origin parameterized by some variable, say c. It is well known that the return map of (2.1), $c \mapsto h(c)$, is analytic if the critical point is nondegenerate. Limit cycles of system (2.1) then correspond to zeros of the *displacement function*, say $d(c) = h(c) - c$. Hence at most k limit cycles can bifurcate from the fine focus. The stability of the origin is clearly dependent on the sign of the first non-zero focal value, and the origin is a nonlinear center if and only if all of the focal values are zero. Consequently, it is the reduced values, or *Lyapunov quantities*, say $L(j)$, that are significant. One needs only to consider the value η_{2k} reduced modulo the ideal $(\eta_2, \eta_4, \ldots, \eta_{2k-2})$ to obtain the Lyapunov quantity $L(k-1)$. To bifurcate limit cycles from the origin, select the coefficients in the Lyapunov quantities such that

$$|L(m)| \ll |L(m+1)| \quad \text{and} \quad L(m)L(m+1) < 0,$$

for $m = 0, 1, \ldots, k-1$. At each stage, the origin reverses stability and a limit cycle bifurcates in a small region of the critical point. If all of these conditions are satisfied, then there are exactly k small-amplitude limit cycles. Conversely, if $L(k) \neq 0$, then at most k limit cycles can bifurcate. Sometimes it is not possible to bifurcate the full complement of limit cycles.

The algorithm for bifurcating small-amplitude limit cycles may be split into the following four steps:

1. computation of the focal values using a mathematical package;
2. reduction of the n-th focal value modulo a Gröbner basis of the ideal generated by the first $n-1$ focal values (or the first $n-1$ Lyapunov quantities);
3. checking that the origin is a center when all of the relevant Lyapunov quantities are zero;
4. bifurcation of the limit cycles by suitable perturbations.

Dongming Wang [34, 35] has developed software to deal with the reduction part of the algorithm for several differential systems. For some systems, the following theorems can be used to prove that the origin is a center.

The Divergence Test. Suppose that the origin of system (1.1) is a critical point of focus type. If

$$\operatorname{div}(\psi \mathbf{X}) = \frac{\partial(\psi P)}{\partial x} + \frac{\partial(\psi Q)}{\partial y} = 0,$$

where $\psi : \Re^2 \to \Re^2$, then the origin is a center.

The Classical Symmetry Argument. Suppose that $\lambda = 0$ in system (2.1) and that either

(i) $p(x,y) = -p(x,-y)$ and $q(x,y) = q(x,-y)$ or

(ii) $p(x,y) = p(-x,y)$ and $q(x,y) = -q(-x,y)$.

Then the origin is a center.

Adapting the classical symmetry argument, it is also possible to prove the following theorem.

Theorem 2.1. *The origin of the system*

$$\dot{x} = y - F(G(x)), \quad \dot{y} = -\frac{G'(x)}{2} H(G(x)),$$

where F and H are polynomials, $G(x) = \int_0^x g(s)ds$ with $g(0) = 0$ and $g(x)\operatorname{sgn}(x) > 0$ for $x \neq 0$, is a center.

To demonstrate the method for bifurcating small-amplitude limit cycles, consider Liénard equations of the form

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{2.3}$$

where $F(x) = a_1x + a_2x^2 + \cdots + a_ux^u$ and $g(x) = x + b_2x^2 + b_3x^3 + \cdots + b_vx^v$. This system has proved very useful in the investigation of limit cycles when showing existence, uniqueness, and hyperbolicity of a limit cycle. In recent years, there have also been many local results; see, for example, [9]. Therefore, it seems sensible to use this class of system to illustrate the method.

The computation of the first three focal values will be given. Write

$$V_k(x, y) = \sum_{i+j=k} V_{i,j}x^iy^j$$

and denote $V_{i,j}$ as being odd or even according to whether i is odd or even and that $V_{i,j}$ is 2-odd or 2-even according to whether j is odd or even, respectively. Solving equation (2.2), it is easily seen that $V_2 = \frac{1}{2}(x^2 + y^2)$ and $\eta_2 = -a_1$. Therefore, set $a_1 = 0$. The odd and even coefficients of V_3 are then given by the two pairs of equations

$$\begin{aligned} 3V_{3,0} - 2V_{1,2} &= b_2, \\ V_{1,2} &= 0 \end{aligned}$$

and

$$\begin{aligned} -V_{2,1} &= a_2, \\ 2V_{2,1} - 3V_{0,3} &= 0, \end{aligned}$$

respectively. Solve the equations to give

$$V_3 = \frac{1}{3}b_2x^3 - a_2x^2y - \frac{2}{3}a_2y^3.$$

Both η_4 and the odd coefficients of V_4 are determined by the equations

$$\begin{aligned} -\eta_4 - V_{3,1} &= a_3, \\ -2\eta_4 + 3V_{3,1} - 3V_{1,3} &= -2a_2b_2, \\ -\eta_4 + V_{1,3} &= 0. \end{aligned}$$

The even coefficients are determined by the equations

$$\begin{aligned} 4V_{4,0} - 2V_{2,2} &= b_3 - 2a_2^2, \\ 2V_{2,2} - 4V_{0,4} &= 0 \end{aligned}$$

and the supplementary condition $V_{2,2} = 0$. In fact, when computing subsequent coefficients for V_{4m}, it is convenient to require that $V_{2m,2m} = 0$. This ensures that there will always be a solution. Solving these equations gives

$$V_4 = \frac{1}{4}(b_3 - 2a_2^2)x^4 - (\eta_4 + a_3)x^3y + \eta_4xy^3$$

and

$$\eta_4 = \frac{1}{8}(2a_2b_2 - 3a_3).$$

Suppose that $\eta_4 = 0$ so that $a_3 = \frac{2}{3}a_2b_2$. It can be checked that the two sets of equations for the coefficients of V_5 give

$$V_5 = \left(\frac{b_4}{5} - \frac{2a_2^2b_2}{3}\right)x^5 + (2a_2^3 - a_4)x^4y + \left(\frac{8a_2^3}{3} - \frac{4a_4}{3} + \frac{2a_2b_3}{3}\right)x^2y^3$$
$$+ \left(\frac{16a_2^3}{15} - \frac{8a_4}{15} - \frac{4a_2b_3}{15}\right)y^5.$$

The coefficients of V_6 may be determined by inserting the extra condition $V_{4,2} + V_{2,4} = 0$. In fact, when computing subsequent even coefficients for V_{4m+2}, the extra condition $V_{2m,2m+2} + V_{2m+2,2m} = 0$, is applied, which guarantees a solution. The polynomial V_6 contains 27 terms and will not be listed here. However, η_6 leads to the Lyapunov quantity

$$L(2) = 6a_2b_4 - 10a_2b_2b_3 + 20a_4b_2 - 15a_5.$$

Lemma 2.1. *The first three Lyapunov quantities for system* (2.3) *are* $L(0) = -a_1$, $L(1) = 2a_2b_2 - 3a_3$, and $L(2) = 6a_2b_4 - 10a_2b_2b_3 + 20a_4b_2 - 15a_5$.

Example. Prove that

(i) there is at most one small-amplitude limit cycle when $\partial F = 3, \partial g = 2$ and

(ii) there are at most two small-amplitude limit cycles when $\partial F = 3, \partial g = 3$,

for system (2.3).

Solutions. (i) Now L(0)=0 if $a_1 = 0$ and $L(1) = 0$ if $a_3 = \frac{2}{3}a_2b_2$. Thus system (2.3) becomes

$$\dot{x} = y - a_2x^2 - \frac{2}{3}a_2b_2x^3, \quad \dot{y} = -x - b_2x^2,$$

and the origin is a center by Theorem 2.1. Therefore, the origin is a fine focus of order one if and only if $a_1 = 0$ and $2a_2b_2 - 3a_3 \neq 0$. The conditions are consistent. Select a_3 and a_1 such that

$$|L(0)| \ll |L(1)| \quad \text{and} \quad L(0)L(1) < 0.$$

The origin reverses stability once and a limit cycle bifurcates. The perturbations are chosen such that the origin reverses stability once and the limit cycles that bifurcate persist.

(ii) Now $L(0) = 0$ if $a_1 = 0$, $L(1) = 0$ if $a_3 = \frac{2}{3}a_2b_2$, and $L(2) = 0$ if $a_2b_2b_3 = 0$. Thus $L(2) = 0$ if

(a) $a_2 = 0$,

(b) $b_3 = 0$, or

(c) $b_2 = 0$.

If condition (a) holds, then $a_3 = 0$ and the origin is a center by the divergence test (div$\mathbf{X} = 0$). If condition (b) holds, then the origin is a center from result (i) above. If condition (c) holds, then $a_3 = 0$ and system (2.3) becomes

$$\dot{x} = y - a_2 x^2, \quad \dot{y} = -x - b_3 x^3,$$

and the origin is a center by the classical symmetry argument. The origin is thus a fine focus of order two if and only if $a_1 = 0$ and $2a_2b_2 - 3a_3 = 0$ but $a_2b_2b_3 \neq 0$. The conditions are consistent. Select b_3, a_3, and a_1 such that

$$|L(1)| \ll |L(2)|, \; L(1)L(2) < 0 \quad \text{and} \quad |L(0)| \ll |L(1)|, \; L(0)L(1) < 0.$$

The origin has changed stability twice, and there are two small-amplitude limit cycles. The perturbations are chosen such that the origin reverses stability twice and the limit cycles that bifurcate persist.

3. Symbolic Computation

Readers can download the following program files from the Web. The MATLAB M-file lists all of the coefficients of the Lyapunov function up to and including degree six terms. The output is also included for completeness. The program was written using MATLAB version 7 and the program files can be downloaded at

http://www.mathworks.com/matlabcentral/fileexchange

under the links "Companion Software for Books" and "Mathematics".

```
% MATLAB Program - Determining the coefficients of the Lyapunov
% function for a quintic Lienard system.

% V3=[V30;V21;V12;V03], V4=[V40;V31;V22;V13;V04;eta4],
% V5=[V50;V41;V32;V23;V14;V05],
% V6=[V60;V51;V42;V33;V24;V15;V06;eta6]
% Symbolic Math toolbox required.
clear all

syms a1 a2 b2 a3 b3 a4 b4 a5 b5;
A=[3 0 -2 0;0 0 1 0;0 -1 0 0;0 2 0 -3];
B=[b2; 0; a2; 0];

V3=A\B

A=[0 -1 0 0  0 -1;0 3 0 -3 0 -2;0 0 0 1 0 -1;4 0 -2 0 0 0;
0 0 2 0 -4 0; 0 0 1 0 0 0];
B=[a3; -2*a2*b2; 0; b3-2*a2^2;0;0];

V4=A\B

A=[5 0 -2 0 0 0;0 0 3 0 -4 0;0 0 0 0 1 0;0 -1 0 0 0 0;0 4 0 -3 0 0;
```

```
0 0 0 2 0 -5];
B=[b4-10*a2^2*b2/3;0;0;a4-2*a2^3;-2*a2*b3;0];

V5=A\B

A=[6 0 -2 0 0 0 0 0;0 0 4 0 -4 0 0 0;0 0 0 0 2 0 -6 0;
0 0 1 0 1 0 0 0;0 -1 0 0 0 0 0 -1;0 5 0 -3 0 0 0 -3;
0 0 0 3 0 -5 0 -3;0 0 0 0 0 1 0 -1];
B=[b5-6*a2*a4-4*a2^2*b2^2/3+8*a2^4;16*a2^4/3+4*a2^2*b3/3-8*a2*a4/3;
0;0;a5-8*a2^3*b2/3;-2*a2*b4+8*a2^3*b2+2*a2*b2*b3-4*a4*b2;
16*a2^3*b2/3+4*a2*b2*b3/3-8*a4*b2/3;0];

V6=A\B

L0=-a1
eta4=V4(6,1)
L1=maple('numer(-3/8*a3+1/4*a2*b2)')
a3=2*a2*b2;
eta6=V6(8,1)
L2=maple('numer(-5/16*a5+1/8*a2*b4-5/24*a2*b2*b3+5/12*a4*b2)')
%End of MATLAB Program
```

```
 V3 =

[  1/3*b2]
[     -a2]
[       0]
[ -2/3*a2]

V4 =

[    1/4*b3-1/2*a2^2]
[ -5/8*a3-1/4*a2*b2]
[                  0]
[ -3/8*a3+1/4*a2*b2]
[                  0]
[ -3/8*a3+1/4*a2*b2]

V5 =

[               1/5*b4-2/3*a2^2*b2]
[                       -a4+2*a2^3]
[                                0]
[      -4/3*a4+8/3*a2^3+2/3*a2*b3]
[                                0]
[ -8/15*a4+16/15*a2^3+4/15*a2*b3]

V6 =
```