Series in Quantitative Finance – Vol. 2

ADVANCED ASSET PRICING THEORY

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Preface

Let us start with a Chinese proverb: "Buried deep inside every good book is a gold mine". It refers to the classical wisdom that the knowledge conveyed by a good book is invaluable. Readers might expect a book like this, written on the subject of finance, to deliver a richer gold reserve. I, of course, cannot make a bold promise of this sort. But, to be fully consistent with the principle of 'no arbitrage', a concept to be discussed in detail in this book, I would rather say that the gold reserved in this book belongs to those who persist in digging. Also, I must apologize in advance to those who expect to see lots of currency symbols like $\$, £, ¥$, etc. The 'currencies' used in this book are a mixture of mathematical symbols like $\sum$ and $\int$ among a bunch of other 'foreign' currencies. To those with a good grasp of undergraduate-level calculus, you are equipped to start digging.

This book introduces modern asset pricing theory. Like many existing books on the subject, it discusses

- how investors make choices in a world of risk and uncertainty;
- how they make decisions on portfolio holdings among a bunch of tradable securities, and how they revise their investment decisions within a given time frame upon the arrival of new information; and
- how security price is determined in an ideal frictionless economy with rational investors.

Special effort has been made to include the study of the information content of security prices. These include

- the search for relevant factors in determining the security price in a frictionless and competitive market environment; and
- an in-depth study of how price reflects the psychological senti-

ment of the investment community in aggregate, which includes, for example, the aggregated psychological attitudes towards risk, and how price reveals the investors' predictions of future aspects of investment, particularly with the fundamentals of the macro-economy.

The book provides rigorously those logical arguments underpinning various theoretical models, and delves into the underlying hypothetical and/or hidden assumptions on the market environment, in addition to those behavior assumptions on the economics agents. In writing this book, my aim is not to sell caught fish to the readers, but to enable readers to fish for themselves, by providing the ideology and appropriate technical skills. In the process of writing this book, several advancements and new theoretical models were made. I hope that this book will facilitate new research breakthroughs, both theoretical and empirical, in this subject.

Special effort has been made to bridge the gap between different approaches to pricing financial securities, namely, the equilibrium approach and the no-arbitrage approach. The former is known to be associated with preference-dependent asset pricing models, while the latter is accompanied by preference-free pricing rules. This is accomplished with insight into the information role of European call options (written on the primitive securities) in revealing the representative agent utility function and in revealing the pricing rule.

With few exceptions, proofs are provided for assertions made in the text without relying heavily on external sources. The self-contained nature of this book makes it an ideal reference on the subject.

Background

Broadly speaking, the theory covered in this book is largely developed under the following two sets of hypothetical assumptions respectively on the market environment and the economic agents (the investors):

- a perfectly competitive market
- utility maximizing rational investors

These assumptions are idealistic since neither assumption can be actually treated as valid in reality. The competitive market assumption is also known as the price-taking behavior assumption on the economic agents —

all investors are small in the sense that an individual investor's trading activity has an ignorable impact on the price of securities. Accordingly, investors' trading activities are undertaken without concern for the reaction of the market to such activities.

The rationality constitutes another set of hypothetical normal behavior assumptions on the economic agents. The rational behavior assumption can be separated into two parts. First, investors make full use of information they possess in decision making. Second, investors set specific goals, summarized by well-defined utility functions, to guide their trading and investment activities. So, rational investors act optimally to maximize their utility functions. An investor's utility function summarizes his preferences over all possible investments.

The equilibrium security prices in such an economy are understood to be set by Adam Smith's *invisible hand.* The concept of equilibrium is probably the most important concept underlying all branches of modern economic theory, including the asset pricing theory to be covered in this book. In short, at equilibrium, prices are set to clear simultaneously the market demand and supply of all tradable securities.

The equilibrium prices serve as aggregators of information. The prices summarize also the non-observable psychological sentiment of the investment community that is represented by the representative agent's utility function. The difficulty in pricing financial securities lies largely in our lack of prior knowledge of the identity of the representative agent and his utility function. Our only hope is, if it exists, to deduce from the observed security prices and other economic variables the non-observable representative agent's utility function.

Keeping these in mind, the effort to seek preference-independent asset pricing models has never ceased in the past five decades. This follows from the original breakthrough made in the early 1960s with the derivation of the Sharpe-Lintner capital asset pricing model (CAPM) named after William Sharpe and John Lintner who delivered the model. Two of the most celebrated breakthroughs towards the derivation of preference-independent asset pricing models are the Black-Scholes option pricing model and the Heath-Jarrow-Morton (HJM) model on the term structure of interest rates.

At this stage, the limitations of these preference-independent models, along with the robustness of the theories with respect to the underlying assumptions on the economic agents and other assumptions made on the information structure and environment of the marketplace, are well recognized. For instance, the CAPM, which is known to constitute an equilib-

rium model in a static economy with mean-preserving-spread risk-averse investors, would in general no longer constitute an equilibrium model in a dynamic setting. The no-arbitrage asset pricing models of Black-Scholes and HJM are also found to be sensitive to the specifications on asset returns, particularly with respect to the assumptions made on the completeness of the market.

The option-based asset pricing theory developed in this book constitutes an important advancement in theoretical asset pricing following in the footsteps of these earlier developments. On one hand, the option pricing rule for European call options written on a specific primitive security can be shown to contain all relevant information for pricing general contingent claims written on the security. On the other hand, the option pricing rule written on the aggregate equity is shown to contain all relevant information for the representative agent's utility function and the fundamentals of the underlying security on which the options are written. With knowledge of the option pricing rule, we are able to bridge the gap between the preference-dependent equilibrium asset pricing models and the preference-independent, option-based asset pricing rule for pricing financial securities.

Organization and Content

The book is organized according to the time frames and the mathematical tools of economics modeling. More specifically, it is divided into three parts:

I. Foundations of asset pricing
II. Discrete-time equilibrium asset pricing
III. Asset pricing in continuous-time

It can be organized alternatively by referring to the pricing of specific financial products such as stocks, bonds, options and so on, or by referring to the different approaches, namely the equilibrium approach and the no-arbitrage approach, to pricing financial securities. Indeed, readers interested in some particular topics, say, the term structure of interest rates, option pricing, equity premium, portfolio choices and so on, should refer to the relevant chapters in the book. For instance, for specific coverage of the no-arbitrage approach to asset pricing, readers are referred to Chapters 1, 2, 5.4, 6.2, 12, 13, 14 and 15. Readers are referred to Chapters 1, 5, 7, 8, 9, 10, 16, 17, 18 and 19 for extensive coverage of equilibrium asset pricing theory, along with derivations and critical assessments of various equilib-

rium asset pricing models. For specific coverage of the term structure of interest rates, turn to Chapters 2.4, 7.3, 9.2, 10.2, 10.9, 15, 18.1 and 19.2. Chapters 2.3, 5.5, 10, 13, 14 and 19 are focused on option pricing and the pricing of financial derivatives. Chapters 4, 7, 8, 16 and 17 have in-depth coverage of risk measurement, portfolio choice and portfolio risk management. Chapters 7, 8, 9, 10, 16, 17, 18 and 19 contain rigorous treatments of the discrete-time recursive utility and the continuous-time stochastic differential utility (SDU), and their corresponding applications to portfolio risk management and asset pricing.

Part I focuses on the micro-economic foundation of asset pricing theory. It consists of five chapters. Chapter 1 is an introduction to the general equilibrium approach to asset pricing along with an illustration of the invisible hand phenomenon, and includes welfare theorems associated with a competitive and frictionless financial market economy. Three basic elements of the financial market, namely, the marketplace, the state of the economy and the economic agents — the investors, are introduced in this chapter. Also included is a discussion on how investors' preferences are formed and how they make their optimal portfolio choice, along with discussions on the absence of arbitrage as a necessary condition for the existence of an optimal portfolio choice for rational investors. The role of the financial market as a mechanism for efficient resource allocation and risk diversification is also studied. As a separate topic, the aggregate characteristics of the financial market, particularly the existence of a representative agent, are studied. This is with the help of a proof of an 'aggregation theorem' that forms a theoretical foundation for many of the most celebrated equilibrium asset pricing models built on the assumption of a representative agent.

Chapter 2 explores the implications of the no-arbitrage condition as part of an equilibrium restriction on security price. It includes detailed discussions on the fundamental theorem of asset pricing in a competitive market environment. Several applications of the fundamental theorem are carried out on option pricing, futures pricing, the term structure of interest rates and the pricing of other derivative securities. It includes derivations of Cox-Ross-Rubinstein's binomial option pricing model, Ho-Lee's model of the term structure of interest rates, and the backward procedure for pricing options and other contingent claims.

Chapter 2 can serve as an introduction to the financial products traded in the marketplace. The financial products covered in this book are largely divided into three categories: stocks, bonds and financial derivatives. It includes portfolios formed by various combinations of securities in these

categories. Some of the most useful pricing relationships among these securities, such as the 'put-call parity' for zero-dividend paying stocks/portfolios and the 'put-call futures parity' for dividend paying stocks/portfolios, are established in this chapter under the no-arbitrage condition.

Chapter 3 is about risk and risk measurement. Various notions of risk measures will be introduced, in addition to discussions on their interrelationships and some in-depth analysis of their relevance to choices made by rational agents.

Chapter 4 studies an investor's portfolio choice problem. The study is carried out for two types of investors: expected utility maximizers and investors whose preferences display mean-preserving-spread (MPS) risk aversion. This chapter is an introduction to behavioral finance and the classical Markowitz mean-variance analysis in a static economy. It also introduces the role of the financial market in risk management. It shows how risk can be diversified through portfolio holdings, and how one can use options and derivative securities to hedge against the downside risk associated with the underlying individual portfolios/securities. It provides theoretical insight into why MPS risk-averse investors tend to invest in few mutual funds (along the lines of Tobin's and Black's mutual-fund separation theorems) among a much wider range of investment opportunities.

Chapter 5 derives the Sharpe-Lintner capital asset pricing model (CAPM) as an equilibrium asset pricing model in an economy of MPS risk-averse agents. The CAPM forms the cornerstone of modern finance theory.

Part II provides an advanced treatment of dynamic asset pricing theory in discrete time. It starts with a coverage of psychological behavior inherited by investors in a world of uncertainty, including when such uncertainty can be partially resolved over a time frame (Chapter 6). It shows how an investor's time preferences, attitude towards risk and attitude towards information acquisition can be characterized by some well-specified utility functions within Epstein-Zin's recursive utility framework. Trading behavior (in terms of portfolio choices) inherited by myopic investors (speculators) and intertemporal recursive-utility maximizing investors (fundamentalists) receive special treatment (Chapters 7 & 8 respectively). It contains detailed derivations of several of the most well known theoretical models in finance (Chapters 7 & 9). Additionally, Chapter 9 includes a newly developed model, known as the shadow-CAPM, as a dynamic analog of the static CAPM that is derived under the MPS risk-averse behavior assumption.

Chapter 10 focuses on pricing contingent claims as applications of the equilibrium model derived in Chapter 9. The pricing rule consists of two components: the term structure of interest rates, and the risk-neutral probability measure. Each component receives special treatment in this chapter. For the term structure of interest rates, there is a derivation of the *long-short parity* in resolving the controversy underpinning the classical expectations hypothesis. It includes also some derivations of an equilibrium model of the term structure of interest rates along with discussions of their potential usefulness. The role of financial options in revealing the unobserved risk-neutral probability measure is formally studied. This includes derivations of a class of equilibrium option pricing formula (written on equities or bonds) in a complete or incomplete market. The derived option pricing formula can accommodate different behavior assumptions of the representative agent; in particular, it emphasizes the discrete-time analogue of the famous Black-Scholes option pricing formula under the expected utility and the log-normal return specifications. The information content of option prices and its usefulness in revealing the unobserved representative agent's utility function receive particular attention.

We then demonstrate the relevance of these theories and theoretical models in addressing several more important and controversial issues documented within the empirical literature: the *equity premium puzzle* and the *short-rate puzzle* underlying the consumption-based CAPM, the *expectations hypothesis* (EH) of the term structure of interest rates, and the *volatility smile* phenomenon associated with the bias of the (discrete-time) Black-Scholes model. Each model's respective effectiveness and advantages in resolving these issues also form a part of this chapter.

Part III develops asset pricing theory in continuous time and contains material ideal for teaching advanced PhD students specializing in theoretical and mathematical finance.

The mathematics in Part III is largely self-contained. Chapter 11 discusses stochastic calculus and stochastic differential equations. It covers the Brownian motion, Poisson process, Lévy jump process and general semimartingale. It includes a specific discussion of the probabilistic properties associated with some of the most used processes in finance. These include the geometric Brownian motion, the geometric Lévy process, the OU process, the CIR's square root process, and the general CEV process. Additionally, Itô's lemma, the Feynman-Kac formula and the Girsanov theorem for stochastic differential equations for general jump-diffusion receive special treatment.

Chapter 12 studies the no-arbitrage approach to asset pricing in continuous time. It extends the results of no-arbitrage asset pricing theory, developed in Chapter 2, to continuous time and contains a rigorous proof of this fundamental theorem in continuous time.

Chapter 13 further develops the Black-Scholes option pricing model as an application of the no-arbitrage theory developed in Chapter 12. We then perform some rigorous mathematical treatments of American options in Chapter 14. The pricing problem is approached in various ways to solve (a) an optimal stopping problem, (b) a free-boundary partial differential equation, (c) an optimal barrier option pricing problem, and (d) a deterministic optimal control problem.

Chapter 15 discusses the no-arbitrage term structure of interest rates. We give equal treatment to models derived using both the classical no-arbitrage approach and the HJM approach. The continuous-time long-short parity analogous to that of discrete-time (derived in Chapter 8) is provided, with which we establish further the equivalence among all three versions of Irving Fisher's classical expectations hypothesis of the term structure of interest rates. The theoretical and empirical merits of the hypothesis are discussed in this chapter.

The remainder of Part III is devoted to equilibrium asset pricing models in continuous-time. Chapter 16 contains a formulation of the SDU of Duffie and Epstein (1992) along with characterizations of the underlying behavior assumptions associated with SDU functions. The sequential choice problem encountered by investors with stochastic differential utilities is studied in Chapter 17. Chapter 18 contains an abstract derivation of a class of intertemporal equilibrium asset pricing models in a jump-diffusion framework with possibly non-expected and non-additive utility functions.

Three applications of equilibrium asset pricing are shown in Chapter 19: (a) equity premium decomposition in extending Breeden's consumption-based capital asset pricing model (C-CAPM) that considers the risk premium for hedging against those small local risks resulting from Brownian uncertainty, and the risk premium for hedging against risks associated with rare jump events, (b) the equilibrium term structure of interest rates in extending the classical CIR model, and (c) the equilibrium option pricing model in the presence of Lévy jumps.

Limitations: This book does not intend to provide an exhaustive coverage of finance and financial economics. Instead, it aims to provide theoretical tools essential to understanding the functioning of the marketplace and the procedures for price-formation in an ideal, frictionless and com-

petitive market environment. Efforts have been made to address issues of financial decision making such as portfolio choices and risk management.

Also, this book does not aim to cover all financial instruments and products as many of the applications discussed in this book are within the following three basic areas: stocks and stock indices, bonds, and derivative securities written on them. Nonetheless, with the tools developed in this book, readers may find it possible to readily tackle other pricing problems that are not specifically covered in this book.

The book touches very little on corporate finance and capital budgeting, market microstructure theory, financial institutions and regulations, and international finance. Nevertheless, the economic agents used in this book could be substituted with consumers, fund managers and firms as well as government agencies to allow the theoretical framework adopted in this book to address issues in other areas of economics and finance.

I chose not to cover recent advances of the theory under Knightian uncertainty based on research. While the idea of distinguishing between risk and uncertainty can be readily accepted by academics, it is, however, beyond the scope of this book to provide an extensive coverage of this particular area of advancement.

Finally, the empirical issues raised in asset pricing theory will be addressed only when it is essential for a better understanding of the theoretical development. The technical aspects of the empirical work will be kept at a basic level.

Readership

These materials were taught in the past at McGill University, the University of Essex, the National University of Singapore, National Taiwan University, Xiamen University, and Fudan University, mainly at the master's and doctorate levels. Part I can be used for an advanced introductory course in the subject for final-year undergraduates, master's and entry-level PhD students. Part II can be used for a one-semester course on dynamic asset pricing (in discrete-time) for postgraduates at both master's and PhD levels. Part III can be used for advanced PhD in continuous-time finance.

I must admit that the book does not target a wide public readership, as it is intended for people who wish to pursue an academic or research career in economics and finance. Researchers and practitioners in the fields may use this book as a reference.

Acknowledgments

This writing project was initiated when I was invited by National Taiwan University to give a lecture series in asset pricing in the fall of 2001. Part of the draft was written during my visit to the National University of Singapore in the spring of 2004. I am grateful to both institutes for organizing the lecture series and for their hospitality.

I have a list of people to thank. The first group are my former teachers in China and Canada. Although there are many, the three who spring to mind are Mr. Zhang Yan-Gao, Professor Chen Zu-Hao and Professor Larry Epstein. Mr. Zhang was my teacher at the No.6 High School of Rong-Cheng, China. I would like to thank him for stimulating my strong interest in mathematics and science during my high school years.

Professor Chen Zu-Hao, my teacher and master's supervisor at Shandong University, China (1983-1985), taught me optimal control theory and variational methods. My first research project on economics modeling was conducted under his supervision when he was appointed Director of the Center for Economics Research of Shandong Province. Without this experience I would have probably followed a very different career path, one that may not have included economics and finance.

Most of all, I am grateful to Professor Larry Epstein, my teacher and PhD supervisor in the University of Toronto, Canada (1987–1992). A large body of my research on asset pricing and decision theory, part of which has been integrated into this book, was initiated when I was Larry's student in Toronto, and largely built upon Larry's earlier work on the subject.

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