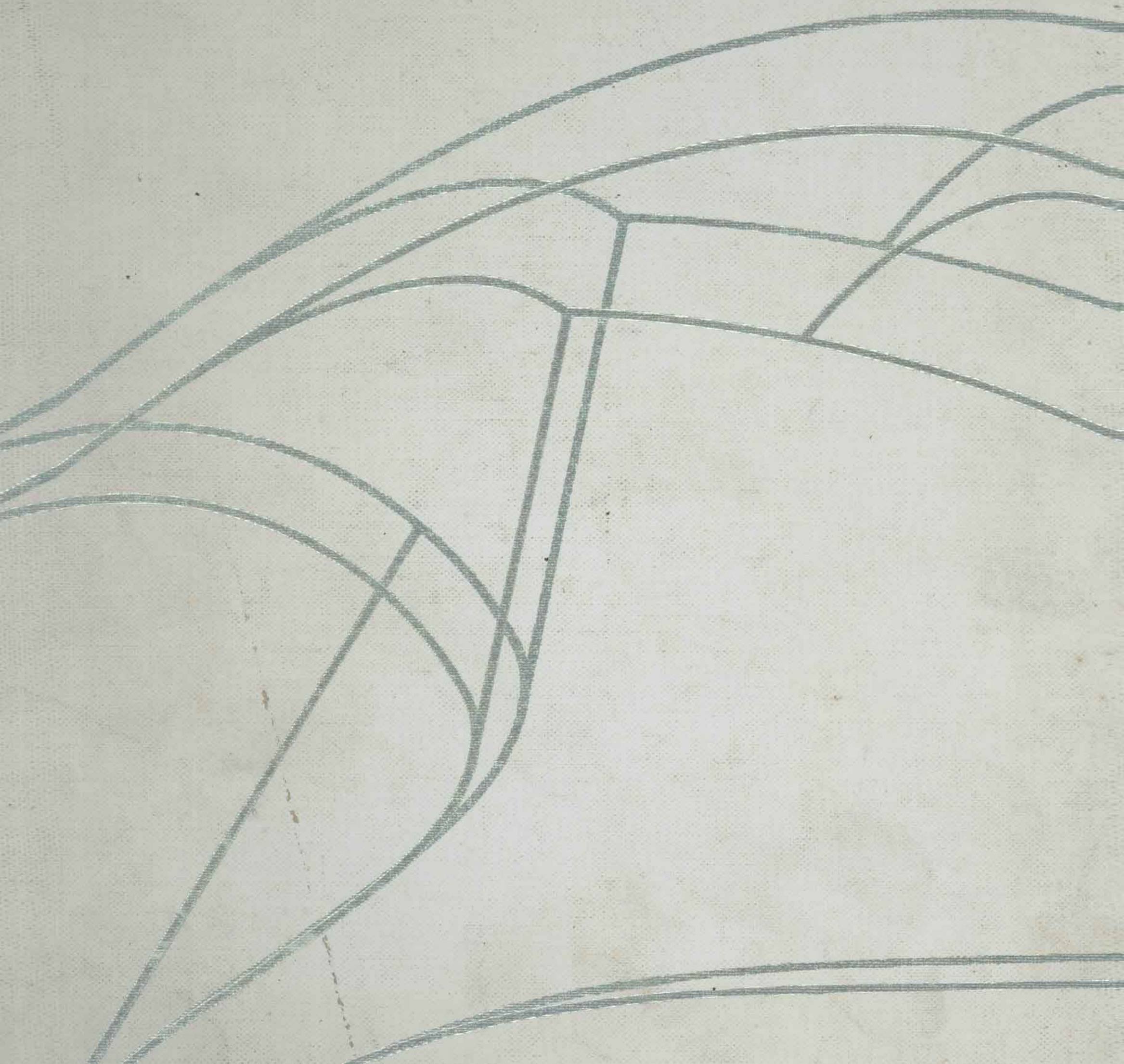
ROBERT T. SEELEY

Calculus of Several Variables



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Calculus of Several Variables
An Introduction

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Calculus of Several Variables
An Introduction

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I. M. SINGER

Massachusetts Institute of Technology

Preface

This book is intended as the "several variables" part of a three- or four-semester calculus course. When he begins, the reader is expected to know the basic theory of functions of one variable, primarily the mean value theorem and the fundamental theorem of calculus. At the end of the book, he should be well prepared for most applications of partial differentiation and line and surface integrals, and for further study in linear algebra and calculus on manifolds.

We work toward an intuitive geometric understanding of vectors, gradients, line integrals, and surface integrals by stressing analytic calculations with a geometric or physical interpretation. Toward the end of Chapter 5 we suggest the unified mathematical framework embracing all these various concepts, laying the groundwork for the general "Stokes' theorem" given in more sophisticated approaches to this subject.

There are almost 400 problems, ranging from trivial exercises to substantial applications (for example, Frenet-Serret formulas, thermodynamics, the planimeter). Most of the difficult problems are broken down into small parts and provided with generous hints, so they need not be reserved for the very best students. There are many more of these problems than any class will have time to do as homework, and there will surely be some left over to serve as classroom examples.

The basic minimum for an introduction to functions of several variables is given in §1.1–1.4, 2.1, 3.1–3.4, 3.6, 4.1–4.3, 5.1, and 5.4. The rest of the material reaches out in various directions: linear algebra, differential geometry of curves, physics, extensions of the fundamental theorem of calculus, and differential forms. The use of these other sections depends, of course, on the time available.

Complete proofs are given for the theorems on differentiation. In integration, the basic result that $\int\int f\,dx\,dy = \int\int f\,dy\,dx$ is clearly formulated, but not proved; see §4.1 for the treatment of this point. Assuming this result, we continue giving complete proofs up to Gauss' theorem and differential forms; the presentation of these two topics is very informal but, we hope, suggestive.

The author gratefully acknowledges the support of Brandeis University, in the form of a sabbatical year, and the further support and hospitality of the Battelle Memorial Institute. He is indebted to S. Lukawecki, H. C. Wiser, and C. R. B. Wright for their useful suggestions, and to Nat Weintraub for his fine editorial work.

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CHAPTER V: FUNCTIONS OF n VARIABLES

Chapter I

Vectors

This chapter provides the setting for the rest of the book. We assume as background a few geometric concepts (such as parallelism, orthogonality, and the Pythagorean theorem, and elementary trigonometry) and develop an abstract algebraic system, called the vector space R^3, in which these concepts have simple algebraic definitions.

§1.1 presents the properties that R^3 shares with the simpler two-dimensional space R^2; familiarity with R^2, though not required, would be useful in reading this section.

§1.2 introduces the cross product.

§1.3 applies vector space methods to analytic geometry.

§1.4 introduces R^n, carrying over the definitions and terminology from R^2 and R^3.

§1.5 introduces the rather abstract idea of *linear independence*. This section, though not essential to the rest of the book, is included partly as preparation for linear algebra (which many students will study sooner or later), and partly because it seems the best way to prove certain basic facts (for example, that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ form a basis if $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \neq 0$). Such facts are used only rarely in the text, so it is possible to omit §1.5 and give intuitive geometric arguments on those few occasions when a reference to §1.5 is made.

We observe a strict separation of powers between "geometry" and "algebra"; geometry suggests and interprets many results, but definitions and proofs are based on elementary algebra. In keeping with this, we are not obliged to prove any statements about the geometric interpretations; they are given only to guide the intuition, not to provide proofs.

1.1 THE VECTOR SPACE R^3

When the Greeks studied solid geometry, they laid the foundations of a remarkably accurate mathematical model of the "physical space" in which stars, planets, rockets, baseballs, electrons, and so on all move about. Solid geometry is still the underlying model for a large part of natural science, but its study has been tremendously simplified by shifting from the purely geometric point of view taken by the Greeks to an algebraic one. From the modern (algebraic) point of view, we define the *vector space* R^3, and establish its various properties by elementary algebraic calculations. To relate the vector space to our intuitive understanding of "physical space" we introduce a coordinate system. Once this is done, all the algebraic results have a more or less obvious geometric meaning.

Definition 1.* The vector space R^3 consists of all ordered triples of real numbers (usually denoted $\mathbf{A} = (a_1,a_2,a_3)$, $\mathbf{B} = (b_1,b_2,b_3)$, etc.), together with the following algebraic operations:

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1\,,\, a_2 + b_2\,,\, a_3 + b_3) \qquad \text{(addition)}$$

$$t\mathbf{A} = (ta_1,ta_2,ta_3), \quad t \text{ any real number} \qquad \text{(scalar multiplication)}$$

$$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(inner product, or dot product)}$$

The members of R^3 are called *points* or *vectors*. The vector (0,0,0) is denoted $\mathbf{0}$. The *length* of a vector $\mathbf{A}$ is

$$|\mathbf{A}| = (\mathbf{A}\cdot\mathbf{A})^{1/2} = \sqrt{a_1^2 + a_2^2 + a_3^2}\,.$$

The numbers a_1, a_2, and a_3 are called *components* or *coordinates* of the vector (a_1,a_2,a_3).

From this definition follow many simple but useful identities, which for convenience we sum up in Theorem 1. You will not find it necessary to memorize all these, since they all reflect familiar properties of numbers.

---

* There is a slight discrepancy in notation and terminology between this definition and Chapter X of *Calculus of One Variable*, which distinguished between points and vectors, denoting points with parentheses (x_0,y_0) and vectors with brackets $[a,b]$. Here the distinction is dropped, and we feel free to think of an ordered triple (a,b,c) either as a point or as a vector. This dual point of view is explained in the discussion following Theorem 1.

Theorem 1. *Let* **A**, **B**, *and* **C** *be any members of* R^3, *and let* t *and* s *be any real numbers. Then*

$$
\left.\begin{array}{ll}
(1) & t(s\mathbf{A}) = (ts)\mathbf{A} \\
(2) & (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\
(3) & t(\mathbf{A}\cdot\mathbf{B}) = (t\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(t\mathbf{B})
\end{array}\right\} \quad (\textit{associative laws})
$$

$$
\left.\begin{array}{ll}
(4) & \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\
(5) & \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}
\end{array}\right\} \quad (\textit{commutative laws})
$$

$$
\left.\begin{array}{ll}
(6) & (t + s)\mathbf{A} = t\mathbf{A} + s\mathbf{A} \\
(7) & t(\mathbf{A} + \mathbf{B}) = t\mathbf{A} + t\mathbf{B} \\
(8) & (\mathbf{A} + \mathbf{B})\cdot\mathbf{C} = (\mathbf{A}\cdot\mathbf{C}) + (\mathbf{B}\cdot\mathbf{C}) \\
(9) & \mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}
\end{array}\right\} \quad (\textit{distributive laws})
$$

$$
(10) \quad 1\cdot\mathbf{A} = \mathbf{A}, \quad 0\mathbf{A} = \mathbf{0}
$$

$$
\left.\begin{array}{ll}
(11) & \mathbf{A} + \mathbf{0} = \mathbf{A} \\
(12) & |\mathbf{A}| = 0 \iff \mathbf{A} = \mathbf{0}
\end{array}\right\} \quad (\textit{laws for the zero vector})
$$

$$
(13) \quad |t\mathbf{A}| = |t|\cdot|\mathbf{A}|.
$$

Proof. Formula (1) follows directly from the commutativity of real numbers; since $a_1 + b_1 = b_1 + a_1$, etc., we have

$$
\begin{aligned}
\mathbf{A} + \mathbf{B} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \\
&= (b_1 + a_1, b_2 + a_2, b_3 + a_3) = \mathbf{B} + \mathbf{A}.
\end{aligned}
$$

The proofs of (2)–(11) follow the same basic pattern; you compute each side, and observe that the corresponding components are equal. Finally, (12) and (13) follow from the formula for the length $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. For example, to prove (12), observe that a sum of squares of real numbers $a_1^2 + a_2^2 + a_3^2$ is zero if and only if each term is zero.

In view of the associative law (2), we can let $\mathbf{A} + \mathbf{B} + \mathbf{C}$ stand for both $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$. More generally, we omit parentheses from any sum of three or more vectors; for example, we simplify $(\mathbf{A} + \mathbf{B}) + (\mathbf{C} + \mathbf{D})$ to $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}$.

Any mathematical system in which formulas (1), (2), (4), (6), (7), (10), and (11) hold is called a *real vector space*. When all of formulas (1)–(13) hold, it is called a *vector space with inner product*. Thus, Theorem 1 states that R^3 *is a vector space with inner product.*

The geometric interpretation of R^3 is based on a rectangular coordinate system. Picture three mutually perpendicular lines intersecting at a given point 0 (the origin), as in Fig. 1. Call these lines the x axis, the y axis, and the z axis, and space the real numbers uniformly along each axis, with zero at the origin. With this picture, to every ordered triple

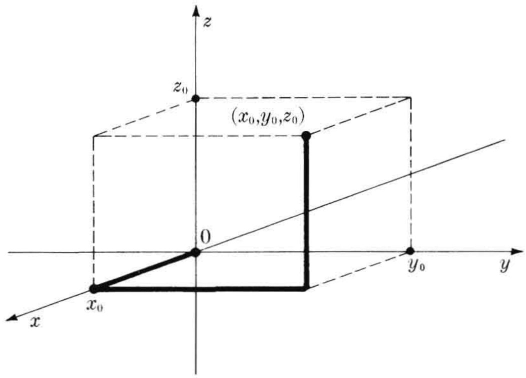


FIGURE 1.1

of real numbers (x_0,y_0,z_0) there corresponds a particular point in space, as shown in Fig. 1. The basic assumption underlying the applications of algebraic methods to concrete geometric and physical problems is that this correspondence is *reversible*; given the coordinate system, every ordered triple corresponds to a unique point in "physical space," and conversely every point corresponds to a unique ordered triple.

Since the members of R^3 are ordered triples, we can think of R^3 geometrically as the points in space. When we have this image in mind, we call the members of R^3 *points*, and generally label them $\mathbf{P} = (x,y,z)$, $\mathbf{P}_0 = (x_0,y_0,z_0)$, and so on. Figure 2 shows that the length $|\mathbf{P}_0|$ is the distance from the origin to the point representing $\mathbf{P}_0$.

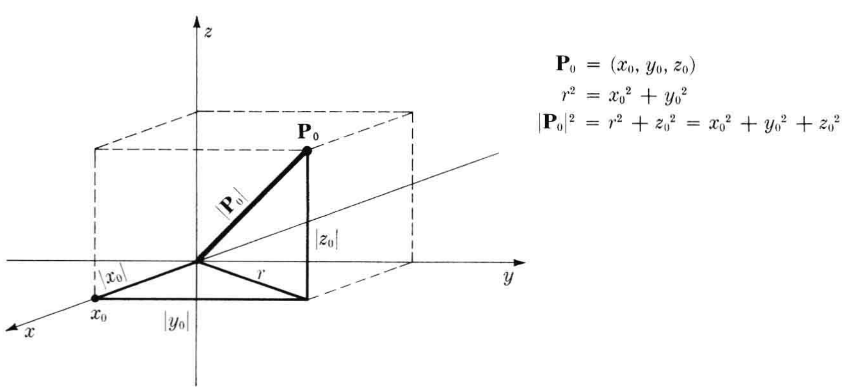
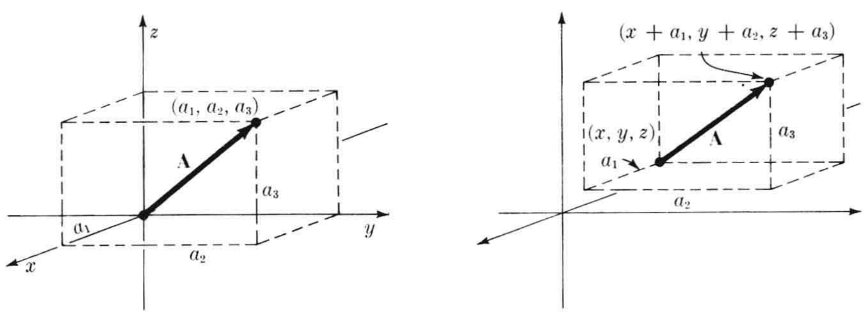


FIGURE 1.2

A second way to represent R^3 is by arrows. If $\mathbf{A} = (a_1,a_2,a_3)$ is any member of R^3, we represent it by an arrow from the origin, as in Fig. 3(a), or more generally by an arrow beginning at any point (x,y,z) and ending at the point $(x + a_1, y + a_2, z + a_3)$, as in Fig. 3(b). (You can think of $\mathbf{A}$ as giving a change in position from the initial point (x,y,z) to the terminal point $(x + a_1, y + a_2, z + a_3)$.) When we have this image in mind, we call the members of R^3 *vectors*. The arrow representing $\mathbf{A}$ can start at any point (and this turns out to be very helpful in visualizing the applications of vector theory), but no matter where it is drawn, it always has the same direction, and the length $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. (We are overlooking the distortions of perspective entailed in drawing lines in three-space on two-dimensional paper.)



(a) (b)

FIGURE 1.3

These two alternate interpretations (point in space, or arrow between two points in space) are closely related. When $\mathbf{A} = (a_1,a_2,a_3)$ is represented by an *arrow beginning at the origin*, as in Fig. 3(a), then the tip of the arrow coincides with the *point* representing (a_1,a_2,a_3). When in doubt as to which interpretation to use, use both, but let the arrow begin at the origin.

Addition has the effect of following one change of position, $\mathbf{A}$, by another, $\mathbf{B}$. Thus $\mathbf{A} + \mathbf{B}$ is represented by the third side of a triangle whose other two sides represent $\mathbf{A}$ and $\mathbf{B}$, as in Fig. 4. Combining this picture with the corresponding one for $\mathbf{B} + \mathbf{A}$ (as in Fig. 5), we find that the commutative law (1) expresses an "obvious fact": $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$ are both represented by the same diagonal of the parallelogram in Fig. 5.

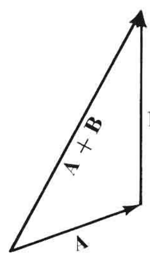


FIGURE 1.4

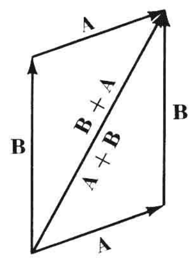


FIGURE 1.5

A familiar theorem of classical geometry states that

$$|\mathbf{A} + \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 \tag{14}$$

if and only if the arrows representing **A** and **B** in Fig. 4 are orthogonal; thus it would be natural to define the vectors themselves to be orthogonal when equation (14) holds. However, this equation can be reduced to a much simpler form if we expand $|\mathbf{A} + \mathbf{B}|^2$ as a dot product:

$$\begin{aligned}
|\mathbf{A} + \mathbf{B}|^2 &= (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) && \text{(by Definition 1)} \\
&= (\mathbf{A} \cdot (\mathbf{A} + \mathbf{B})) + (\mathbf{B} \cdot (\mathbf{A} + \mathbf{B})) && \text{(by (8))} \\
&= (\mathbf{A} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{B}) + (\mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B}) && \text{(by (9))} \\
&= |\mathbf{A}|^2 + 2(\mathbf{A} \cdot \mathbf{B}) + |\mathbf{B}|^2 && \text{(by (2) and (5).)}
\end{aligned}$$

Hence $|\mathbf{A} + \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2$ if and only if $\mathbf{A} \cdot \mathbf{B} = 0$. This motivates

Definition 2. **A** and **B** are called *orthogonal* if and only if $\mathbf{A} \cdot \mathbf{B} = 0$.

Turning next to the *scalar product* $t\mathbf{A}$ of a real number t and a vector **A**, we find that if $t > 0$, then $t\mathbf{A}$ is a change of position in the same direction as **A** but t times as far, while if $t < 0$, then $t\mathbf{A}$ is in the opposite direction from **A** but $|t|$ times as far (Fig. 6). In any case, the vector $t\mathbf{A}$ appears to be *parallel* to **A**; this motivates

Definition 3. Two vectors are *parallel* if and only if one is a scalar multiple of the other. In other words, **A** and **B** are parallel if and only if either $\mathbf{A} = t\mathbf{B}$ for some real t or $\mathbf{B} = t\mathbf{A}$ for some real t.

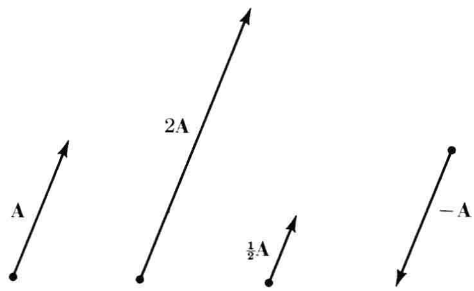


FIGURE 1.6

Combining addition and scalar multiplication leads to interesting results. Given two vectors **A** and **B**, with **B** $\neq$ **0**, look at the vectors of the form

$$\mathbf{A} + t\mathbf{B}, \tag{15}$$

where t varies over all real numbers. With $t = 0$ we get **A**; with $t = 1$ we get **A** + **B**; with $t = -1$ we get **A** + (-1)**B**, which is usually written simply **A** − **B** and called the *difference* of **A** and **B**. In general, as t varies over the real numbers, the vectors (15) generate a *line* as sketched in Fig. 7, called the line through **A** in the direction **B**. (It must be assumed that **B** $\neq$ **0**, for otherwise (15) gives only the point **A**, not a whole line.) Visualize the line as consisting of all the points that represent **A** + t**B** for various t or, equivalently, as the tips of the arrows representing **A** + t**B**, when the arrows begin at the origin.

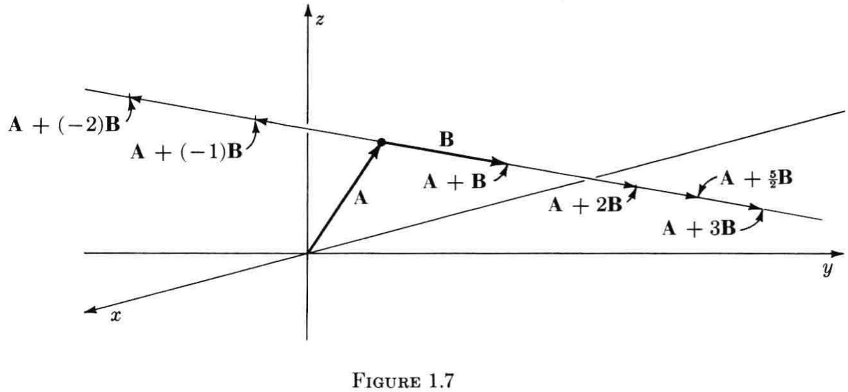


FIGURE 1.7

Figure 7 shows the difference $\mathbf{A} - \mathbf{B}$ as the sum of the vectors $\mathbf{A}$ and $(-1)\mathbf{B}$. Figure 8 shows another useful representation; when $\mathbf{A}$ and $\mathbf{B}$ are drawn from a common initial point, then $\mathbf{A} - \mathbf{B}$ is represented by an arrow from the tip of $\mathbf{B}$ to the tip of $\mathbf{A}$. The resulting triangle illustrates the identity

$$\mathbf{B} + (\mathbf{A} - \mathbf{B}) = \mathbf{A}.$$

Returning to the line given by (15), we can collect an unexpected dividend by computing the distance from the origin to the line. By definition, this distance is the minimum of $|\mathbf{A} + t\mathbf{B}|$ as t varies over all real numbers. To find the minimum, expand $|\mathbf{A} + t\mathbf{B}|^2$ as a dot product (just like the calculation preceding Definition 2):

$$|\mathbf{A} + t\mathbf{B}|^2 = \mathbf{A}\cdot\mathbf{A} + 2t(\mathbf{A}\cdot\mathbf{B}) + t^2\mathbf{B}\cdot\mathbf{B}. \tag{16}$$

On the right in (16) is a quadratic in t (since $|\mathbf{B}| \neq 0$), and its minimum is easily found to occur at $t = -\mathbf{A}\cdot\mathbf{B}/\mathbf{B}\cdot\mathbf{B}$. Putting this value of t in each side of (16), we find the square of the distance from the line to the origin to be

$$\begin{aligned}\left|\mathbf{A} - \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}}\mathbf{B}\right|^2 &= \mathbf{A}\cdot\mathbf{A} - 2\frac{(\mathbf{A}\cdot\mathbf{B})^2}{\mathbf{B}\cdot\mathbf{B}} + \frac{(\mathbf{A}\cdot\mathbf{B})^2}{\mathbf{B}\cdot\mathbf{B}} \\ &= |\mathbf{A}|^2 - \frac{(\mathbf{A}\cdot\mathbf{B})^2}{|\mathbf{B}|^2} \\ &= \frac{1}{|\mathbf{B}|^2}\left(|\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A}\cdot\mathbf{B})^2\right).\end{aligned} \tag{17}$$

Since the number in (17) is a square, it cannot be negative; hence $(\mathbf{A}\cdot\mathbf{B})^2 \leq |\mathbf{A}|^2\cdot|\mathbf{B}|^2$. Taking square roots, we collect our dividend, the *Schwarz inequality*:

$$|(\mathbf{A}\cdot\mathbf{B})| \leq |\mathbf{A}|\cdot|\mathbf{B}|. \tag{18}$$

This derivation assumed $\mathbf{B} \neq \mathbf{0}$, but (18) is obviously true when $\mathbf{B} = \mathbf{0}$ as well, since both sides reduce to zero in that case.

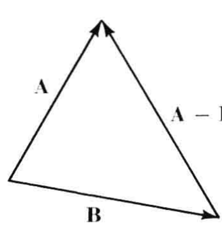


FIGURE 1.8

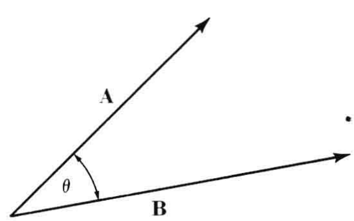


FIGURE 1.9

The Schwarz inequality implies, in turn, the *triangle inequality*:

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|. \tag{19}$$

The proof of (19) is straightforward; from (16) we have already found that

$$|\mathbf{A} + \mathbf{B}|^2 = |\mathbf{A}|^2 + 2(\mathbf{A}\cdot\mathbf{B}) + |\mathbf{B}|^2; \tag{20}$$

hence

$$\begin{aligned} |\mathbf{A} + \mathbf{B}|^2 &\leq |\mathbf{A}|^2 + 2|(\mathbf{A}\cdot\mathbf{B})| + |\mathbf{B}|^2 \\ &\leq |\mathbf{A}|^2 + 2|\mathbf{A}|\cdot|\mathbf{B}| + |\mathbf{B}|^2 \qquad \text{(by (18))} \\ &= (|\mathbf{A}| + |\mathbf{B}|)^2, \end{aligned}$$

and the triangle inequality (19) follows by taking square roots. Figure 4 interprets this inequality as a familiar principle of Euclidean geometry, namely, any side of a triangle is less than or equal to the sum of the other two sides. (Hence the name "triangle inequality.")

The geometric interpretation of the dot product is based on the Schwarz inequality. When $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$, we can rewrite (18) as

$$-1 \leq \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\cdot|\mathbf{B}|} \leq 1.$$

Hence $\arccos\left(\dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\cdot|\mathbf{B}|}\right)$ is defined; namely, it is the number θ such that $0 \leq \theta \leq \pi$ and

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\cdot|\mathbf{B}|}. \tag{21}$$

This is called the *angle between* **A** *and* **B** (see Fig. 9). Notice that (21) is consistent with the definition of orthogonal vectors, since for $0 \leq \theta \leq \pi$

$$\theta = \frac{\pi}{2} \quad \Leftrightarrow \quad \cos\theta = 0 \quad \Leftrightarrow \quad \mathbf{A}\cdot\mathbf{B} = 0.$$

It is also consistent with the definition of parallel vectors, and with the law of cosines, as shown in the problems below. When (21) is multiplied out, it gives

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\cdot|\mathbf{B}|\cos\theta,$$

which is the geometric interpretation we were looking for: the dot product of two vectors is the product of the lengths times the cosine of the angle between the vectors.