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Mathematical Foundations of Supersymmetry

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Preface

Supersymmetry was discovered by physicists in the 1970s. The mathematical treatment of it began much later and grew out of the works notably of Berezin, Kostant, Leites, Manin, Bernstein, Freed, Deligne, Morgan, Varadarajan and others. These works are all in what one may call the differential category and contain many additional references to the subject.

This monograph has grown out of the desire to present a moderately brief and focussed account of the mathematical foundations of supersymmetry both in the differential and algebraic categories. Our view is that supergeometry and super Lie theory are beautiful areas and deserve much attention.

Our intention was not to write an encyclopedic treatment of supersymmetry but to supply only the foundational material that will allow the reader to penetrate the more advanced papers in the wide literature on this subject. For this reason we do not treat the differential and symplectic supergeometry and we are unable to give a comprehensive treatment of the representation theory of Lie supergroups and Lie superalgebras, which can be found in more advanced papers by Kac, Serganova, Penkov, Duflo, Cassinelli et al. and so on.

Our work is primarily directed to second or third year graduate students who have taken a one year graduate course in algebra and a beginning course in Lie groups and Lie algebras. We have provided a discussion without proofs of the classical theory, which will serve as a departure point for our supergeometric treatment. Our book can very well be used as a one-semester course or a participating seminar on supersymmetry, directed to second and third year graduate students.

The language used in this monograph is that of the functor of points. Since this language is not always familiar even to second-year graduate students we have attempted to explain it even at the level of classical geometry. Apart from being the most natural medium for understanding supergeometry, it is also, remarkably enough, the language closest to the physicists' method of working with supersymmetry.

We wish to thank professor V. S. Varadarajan for introducing us to this beautiful part of mathematics. He has truly inspired us through his insight and deep understanding of the subject. We also wish to thank Dr. L. Balduzzi, Prof. G. Cassinelli, Prof. A. Cattaneo, Prof. M. Duflo, Prof. F. Gavarini, Prof. A. Kresch, Prof. M. A. Lledo, Prof. L. Migliorini, Prof. I. M. Musson, Prof. V. Ovsienko, Dr. E. Petracci, Prof. A. Vistoli and Prof. A. Zubkov for helpful remarks. We also want to thank the UCLA Department of Mathematics, the Dipartimento di Matematica, Università di Bologna, and the Dipartimento di Fisica, Università di Genova, for support and hospitality during the realization of this work.

Introduction

Supersymmetry (SUSY) is the machinery mathematicians and physicists have developed to treat two types of elementary particles, *bosons* and *fermions*, on the same footing. Supergeometry is the geometric basis for supersymmetry; it was first discovered and studied by physicists, Wess and Zumino [80], Salam and Strathdee [65] (among others), in the early 1970s. Today supergeometry plays an important role in high energy physics. The objects in super geometry generalize the concept of smooth manifolds and algebraic schemes to include anticommuting coordinates. As a result, we employ the techniques from algebraic geometry to study such objects, namely A. Grothendieck's theory of schemes.

Fermions include all of the material world; they are the building blocks of atoms. Fermions do not like each other. This is in essence the Pauli exclusion principle which states that two electrons cannot occupy the same quantum mechanical state at the same time. Bosons, on the other hand, can occupy the same state at the same time.

Instead of looking at equations that simply describe either bosons or fermions separately, supersymmetry seeks out a description of both simultaneously. Transitions between fermions and bosons require that we allow transformations between the commuting and anticommuting coordinates. Such transitions are called supersymmetries.

In classical Minkowski space, physicists classify elementary particles by their mass and spin. Einstein's special theory of relativity requires that physical theories must be invariant under the Poincaré group. Since observable operators (e.g. Hamiltonians) must commute with this action, the classification corresponds to finding unitary representations of the Poincaré group. In the SUSY world, this means that mathematicians are interested in unitary representations of the super Poincaré group. A "super" representation gives a "multiplet" of ordinary particles which include both fermions and bosons.

Up to this point, there have been no colliders that can produce the energy required to physically expose supersymmetry. However, the Large Hadron Collider (LHC) in CERN (Geneva, Switzerland) became operational in 2007. Physicists are planning proton–proton and proton–antiproton collisions which will produce energies high enough where it is believed supersymmetry can be seen. Such a discovery will solidify supersymmetry as the most viable path to a unified theory of all known forces. Even before the boson–fermion symmetry which SUSY presupposes is proved to be physical fact, the mathematics behind the theory is quite remarkable. The concept that space is an object built out of local pieces with specific local descriptions has evolved through many centuries of mathematical thought. Euclidean and non-Euclidean geometry, Riemann surfaces, differentiable manifolds, complex manifolds, algebraic varieties, and so on represent various stages of this concept. In Alexander Grothendieck's theory of schemes, we find a single structure that encompasses all previous ideas of space. However, the fact that conventional descriptions of space will fail at very small distances (Planck length) has been the driving force behind the discoveries of unconventional models of space that are rich enough to portray the quantum fluctuations of space at these unimaginably small distances. Supergeometry is perhaps the most highly developed of these theories; it provides a surprising application and continuation of the Grothendieck theory and opens up large vistas. One should not think of it as a mere generalization of classical geometry, but as a deep continuation of the idea of space and its geometric structure.

Out of the first supergeometric objects constructed by the pioneering physicists came mathematical models of superanalysis and supermanifolds independently by F. A. Berezin [10], B. Kostant [49], D. A. Leites [53], and De Witt [25]. The idea to treat a supermanifold as a ringed space with a sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded algebras was introduced in these early works. Later, Bernstein [22] and Leites [53] used techniques from algebraic geometry to deepen the study of supersymmetry. In particular, Bernstein and Leites accented the functor of points approach from Grothendieck's theory of schemes. Interest in SUSY has grown in the past decade, and most recently works by V. S. Varadarajan [76] and others have continued exploration of this beautiful area of physics and mathematics and have inspired this work. Given the interest and the number of people who have contributed greatly to this field from various perspectives, it is impossible to give a fair and accurate account of all the works related to ours. We have nevertheless made an attempt and have provided bibliographical references at the end of each chapter, pointing out the main papers that have inspired our work. We apologize for any involuntary omissions.

In our exposition of mathematical SUSY, we use the language of T-points to build supermanifolds up from their foundations in $\mathbb{Z}/2\mathbb{Z}$-graded linear algebra (superalgebra). The following is a brief description of our work.

In Chapter 1 we begin by studying $\mathbb{Z}/2\mathbb{Z}$-graded linear objects. We define super vector spaces and superalgebras, then generalize some classical results and ideas from linear algebra to the super setting. For example, we define a super Lie algebra, discuss supermatrices, and formulate the super trace and determinant (the Berezinian). We also discuss the Poincaré–Birkhoff–Witt theorem in full detail.

In Chapter 2 we provide a brief account of classical sheaf theory with a section dedicated to schemes. This is meant to be an introductory chapter on this subject and the advanced reader may very well skip it.

In Chapter 3 we introduce the most basic geometric structure: a superspace. We present some general properties of superspaces which lead into two key examples of superspaces, supermanifolds and superschemes. Here we also introduce the notion of T-points which allows us to treat our geometric objects as functors; it is a fundamental tool to gain geometric intuition in supergeometry.

Chapters 4–9 lay down the full foundations of C^∞-supermanifolds over $\mathbb{R}$. In Chapter 4, we give a complete proof of foundational results like the chart theorem and the correspondence between morphisms of supermanifolds and morphisms of the superalgebras of their global sections. In Chapter 5 we discuss the local structure of morphisms proving the analog of the inverse function, submersion and immersion theorems. In Chapter 6 we prove the local and global Frobenius theorem on supermanifolds. In Chapters 7 and 8 we give special attention to super Lie groups and their associated Lie algebras, as well as look at how group actions translate infinitesimally. We then use infinitesimal actions and their characterizations to build the super Lie subgroup, subalgebra correspondence. Finally in Chapter 9 we discuss quotients of Lie supergroups.

Chapters 10, 11 expand upon the notion of a superscheme which we introduce in Chapter 3. We immediately adopt the language of T-points and give criteria for representability: in supersymmetry it is often most convenient to describe an object functorially, and then show that it is representable. We explicitly construct the Grassmannian functorially, then use the representability criterion to show that it is a superscheme. Chapter 10 concludes with an examination of the infinitesimal theory of superschemes.

We continue this exploration in Chapter 11 from the point of view of algebraic supergroups and their Lie algebras. We discuss the linear representations of affine algebraic supergroups; in particular we show that all affine supergroups are realized as subgroups of the general linear supergroup.

We have made an effort to make this work self-contained and suggest that the reader begins with Chapters 1–3, but Chapters 4–9 and Chapters 10–11 are somewhat disjoint and may be read independently.

Contents

1

$\mathbb{Z}/2\mathbb{Z}$-graded linear algebra

The theory of manifolds and algebraic geometry are ultimately based on linear algebra. Similarly the theory of supermanifolds needs super linear algebra, which is linear algebra in which vector spaces are replaced by vector spaces with a $\mathbb{Z}/2\mathbb{Z}$-grading, namely, super *vector spaces*. The basic idea is to develop the theory along the same lines as the usual theory, adding modifications whenever necessary. We therefore first build the foundations of linear algebra in the super context. This is an important starting point as we later build super geometric objects from sheaves of super linear spaces.

Let us fix a ground field k, $\mathrm{char}(k) \neq 2, 3$.

1.1 Super vector spaces and superalgebras

Definition 1.1.1. A *super vector space* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$V = V_0 \oplus V_1$$

where elements of V_0 are called "even" and elements of V_1 are called "odd".

Definition 1.1.2. The *parity* of $v \in V$, denoted by $p(v)$ or $|v|$, is defined only on non-zero *homogeneous* elements, that is elements of either V_0 or V_1:

$$p(v) = |v| = \begin{cases} 0 \text{ if } v \in V_0, \\ 1 \text{ if } v \in V_1. \end{cases}$$

Since any element may be expressed as the sum of homogeneous elements, it suffices to consider only homogeneous elements in the statement of definitions, theorems, and proofs.

Definition 1.1.3. The *superdimension* of a super vector space V is the pair (p, q) where $\dim(V_0) = p$ and $\dim(V_1) = q$ as ordinary vector spaces. We simply write $\dim(V) = p|q$.

From now on we will simply refer to the superdimension as the dimension when it is clear that we are working with super vector spaces. If $\dim(V) = p|q$, then we can find a basis $\{e_1, \dots, e_p\}$ of V_0 and a basis $\{\epsilon_1, \dots, \epsilon_q\}$ of V_1 so that V is canonically isomorphic to the free k-module generated by the $\{e_1, \dots, e_p, \epsilon_1, \dots, \epsilon_q\}$. We denote this k-module by $k^{p|q}$ and we will call $(e_1, \dots, e_p, \epsilon_1, \dots, \epsilon_q)$ the *canonical basis* of $k^{p|q}$. The (e_i) form a basis of $k^p = k_0^{p|q}$ and the (ϵ_j) form a basis for $k^q = k_1^{p|q}$.

Definition 1.1.4. A *morphism* from a super vector space V to a super vector space W is a linear map from V to W preserving the $\mathbb{Z}/2\mathbb{Z}$-grading. Let $\mathrm{Hom}(V, W)$ denote the vector space of morphisms $V \to W$.

Thus we have formed the category[1] of super vector spaces that we denote by (smod). It is important to note that the category of super vector spaces also admits an "inner Hom", which we denote by $\underline{\mathrm{Hom}}(V, W)$; for super vector spaces V, W, $\underline{\mathrm{Hom}}(V, W)$ consists of *all* linear maps from V to W; it is made into a super vector space itself by the following definitions:

$$\underline{\mathrm{Hom}}(V, W)_0 = \{T : V \to W \mid T \text{ preserves parity}\} \quad (= \mathrm{Hom}(V, W));$$

$$\underline{\mathrm{Hom}}(V, W)_1 = \{T : V \to W \mid T \text{ reverses parity}\}.$$

If $V = k^{m|n}$, $W = k^{p|q}$ we have, in the canonical basis (e_i, ϵ_j):

$$\underline{\mathrm{Hom}}(V, W)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \quad \text{and} \quad \underline{\mathrm{Hom}}(V, W)_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where A, B, C, D are respectively $(p \times m)$, $(p \times n)$, $(q \times m)$, $(q \times n)$-matrices with entries in k.

In the category of super vector spaces we have the *parity reversing functor* $\Pi(V \to \Pi V)$ defined by

$$(\Pi V)_0 = V_1, \quad (\Pi V)_1 = V_0.$$

The category of super vector spaces admits tensor products: for super vector spaces V, W, $V \otimes W$ is given the $\mathbb{Z}/2\mathbb{Z}$-grading as follows:

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1),$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

The assignment $V, W \mapsto V \otimes W$ is additive and exact in each variable as in the ordinary vector space category. The object k functions as a unit element with respect to tensor multiplication $\otimes$; and tensor multiplication is associative, i.e., the two products $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$ are naturally isomorphic. Moreover, $V \otimes W \cong W \otimes V$ by the *commutativity map*

$$c_{V,W} : V \otimes W \to W \otimes V$$

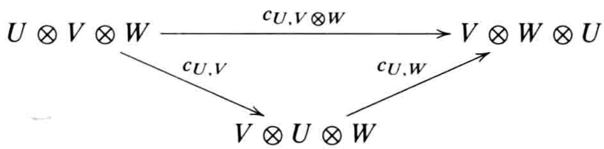
where $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.

The significance of this definition is as follows. If we are working with the category of vector spaces, the commutativity isomorphism takes $v \otimes w$ to $w \otimes v$. In super linear algebra we have to add the sign factor in front. This is a special case of the general

[1] We refer the reader not accustomed to category language to Appendix B.1.

principle called the "sign rule" that one finds in some physics and mathematics literature. The principle says that in making definitions and proving theorems, the transition from the usual theory to the super theory is often made by just simply following this principle, which introduces a sign factor whenever one reverses the order of two odd elements. The functoriality underlying the constructions makes sure that the definitions are all consistent.

The commutativity isomorphism satisfies the so-called *hexagon diagram*:



where, if we had not suppressed the arrows of the associativity morphisms, the diagram would have the shape of a hexagon.

The definition of the commutativity isomorphism, also informally referred to as the sign rule, has the following very important consequence. If $V_1, \dots, V_n$ are super vector spaces and σ and τ are two permutations of n elements, no matter how we compose associativity and commutativity morphisms, we always obtain the same isomorphism from $V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}$ to $V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)}$ namely:

$$V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} \to V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)},$$

$$v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mapsto (-1)^N v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}$$

where N is the number of pairs of indices i, j such that v_i and v_j are odd and $\sigma^{-1}(i) < \sigma^{-1}(j)$ with $\tau^{-1}(i) > \tau^{-1}(j)$.

The *dual*, V^*, of V is defined as

$$V^* := \underline{\mathrm{Hom}}(V, k).$$

Notice that, if V is even, that is $V = V_0$, we have V^* is the ordinary dual of V, consisting of all even morphisms $V \to k$. If V is odd, that is $V = V_1$, then V^* is also an odd vector space and consists of all odd morphisms $V^1 \to k$. This is because any morphism from V_1 to $k = k^{1|0}$ is necessarily odd since it sends odd vectors into even ones.

The category of super vector spaces thus becomes what is known as a *tensor category with inner Hom and dual*. We start by recalling the universal property of the tensor product.

Proposition 1.1.5. *Let V and W be two super vector spaces and f a bilinear map of $V \times W$ into a third super vector space Z. Then there exists a unique morphism $g\colon V \otimes W \to Z$ such that*

$$g(v \otimes w) = f(v, w) \quad (v \in V,\ w \in W).$$

Proof. See [51], Ch. XVI. □

Remark 1.1.6. The object $V^{\otimes n} = V \otimes \cdots \otimes V$ (n times) for a super vector space V is perfectly well defined. We can extend this notion to make sense of $V^{\otimes n|m}$ via the parity reversing functor Π. Define

$$V^{n|m} := \underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \times \underbrace{\Pi(V) \times \Pi(V) \times \cdots \times \Pi(V)}_{m \text{ times}},$$

from which the definition of $V^{\otimes n|m}$ follows by the universal property. In other words, we have:

$$V^{\otimes n|m} := V \otimes V \otimes \cdots \otimes V \otimes \Pi(V) \otimes \Pi(V) \otimes \cdots \otimes \Pi(V)$$

where the parity is coming from the tensor product.

In the ordinary setting, an algebra is a vector space A with a multiplication which is bilinear. We may therefore think of it as a vector space A together with a linear map $A \otimes A \to A$. We now define a superalgebra in the same way:

Definition 1.1.7. A *superalgebra* is a super vector space A together with a multiplication morphism $\tau \colon A \otimes A \to A$.

We then say that a superalgebra A is *(super)commutative* if

$$\tau \circ c_{A,A} = \tau,$$

that is, if the product of homogeneous elements obeys the rule

$$ab = (-1)^{|a||b|} ba.$$

This is an example of the sign rule mentioned earlier. Note that the signs do not appear in the definition; this is the advantage of the categorical view point which suppresses signs and therefore streamlines the theory.

Similarly we say that A is *associative* if

$$\tau \circ \tau \otimes \mathrm{id} = \tau \circ \mathrm{id} \otimes \tau$$

on $A \otimes A \otimes A$. In other words if $(ab)c = a(bc)$. We also say that A has a *unit* if there is an even element 1 so that

$$\tau(1 \otimes a) = \tau(a \otimes 1) = a$$

for all $a \in A$, that is if $a \cdot 1 = 1 \cdot a = a$.

The tensor product $A \otimes B$ of two superalgebras A and B is again a superalgebra, with multiplication defined as

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd).$$

As an example of associative superalgebra we are going to define the tensor superalgebra.

Definition 1.1.8. Let V be a super vector space. We define *tensor superalgebra* to be the super vector space

$$T(V)=\bigoplus_{n\geq 0} V^{\otimes n}, \quad T(V)_0=\bigoplus_{n \text{ even}} V^{\otimes n}, \quad T(V)_1=\bigoplus_{n \text{ odd}} V^{\otimes n},$$

together with the product defined, as usual, via the ordinary bilinear map $\phi_{r,s}\colon V^{\otimes r}\times V^{\otimes s}\to V^{\otimes(r+s)}$,

$$\phi_{r,s}(v_{i_1}\otimes\cdots\otimes v_{i_r}, w_{j_1}\otimes\cdots\otimes w_{j_s})=v_{i_1}\otimes\cdots\otimes v_{i_r}\otimes w_{j_1}\cdots\otimes w_{j_s}.$$

One can check that $T(V)$ is a well-defined associative superalgebra with unit, which is noncommutative except when V is even and one-dimensional.

From now on we will assume that all superalgebras are associative and with unit unless specified. Moreover we shall denote the category of commutative superalgebras by (salg).

If we take a super vector space and mod out the odd part, we obtain a classical (that is, purely even) vector space. In a superalgebra the corresponding object is defined by taking the quotient by the ideal generated by the odd elements. This allows one to always refer back to the classical setting.

We denote by J_A the ideal in the commutative superalgebra A generated by the odd elements in A.

Example 1.1.9 (Grassmann coordinates). Let

$$A=k[t_1,\dots,t_p,\theta_1,\dots,\theta_q]$$

where the $t_1,\dots,t_p$ are ordinary indeterminates and the $\theta_1,\dots,\theta_q$ are *odd indeterminates*, i.e., they behave like Grassmann coordinates:

$$\theta_i\theta_j=-\theta_j\theta_i.$$

(This of course implies that $\theta_i^2=0$ for all i.) In other words we can view A as the ordinary tensor product $k[t_1,\dots,t_p]\otimes\wedge(\theta_1,\dots,\theta_q)$, where $\wedge(\theta_1,\dots,\theta_q)$ is the exterior algebra generated by $\theta_1,\dots,\theta_q$.

As one can readily check, A is a supercommutative algebra. In fact,

$$A_0=\left\{f_0+\textstyle\sum_{|I| \text{ even}} f_I\theta_I \mid I=\{i_1<\cdots<i_r\}\right\}$$

where $\theta_I=\theta_{i_1}\theta_{i_2}\dots\theta_{i_r}$, $|I|=r$ and $f_0, f_I\in k[t_1,\dots,t_p]$, and

$$A_1=\left\{\textstyle\sum_{|J| \text{ odd}} f_J\theta_J \mid J=\{j_1<\cdots<j_s\}\right\}.$$

Note that although the $\{\theta_j\}\in A_1$, there are plenty of nilpotents in A_0; take for example $\theta_1\theta_2\in A_0$.

This example is important since any finitely generated commutative superalgebra is isomorphic to a quotient of the algebra A by a homogeneous ideal.

As one can readily check, $J_A=(\theta_1,\dots,\theta_q)$ and $A/J_A\cong k[t_1,\dots,t_p]$.