A Short Course on Banach Space Theory

N. L. Carothers

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A SHORT COURSE ON BANACH SPACE THEORY

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Preface

These are notes for a graduate topics course offered on several occasions to a rather diverse group of doctoral students at Bowling Green State University. An earlier version of these notes was available through my Web pages for some time and, judging from the e-mail I've received, has found its way into the hands of more than a few readers around the world. Offering them in their current form seemed like the natural thing to do.

Although my primary purpose for the course was to train one or two students to begin doing research in Banach space theory, I felt obliged to present the material as a series of compartmentalized topics, at least some of which might appeal to the nonspecialist. I managed to cover enough topics to suit my purposes and, in the end, assembled a reasonable survey of at least the rudimentary tricks of the trade.

As a prerequisite, the students all had a two-semester course in real analysis that included abstract measure theory along with an introduction to functional analysis. While abstract measure theory is only truly needed in the final chapter, elementary facts from functional analysis, such as the Hahn–Banach theorem, the Open Mapping theorem, and so on, are needed throughout. Chapter 2, "Preliminaries," offers a brief summary of several key ideas from functional analysis, but it is far from self-contained. This chapter also features a large set of exercises I used as the basis for additional review, when necessary. A modest background in topology is also helpful but, because many of my students needed review here, I included a brief appendix containing most of the essential facts.

I make no claims of originality here. In fact, the presentation borrows heavily from several well-known sources. I tried my best to document these sources fully in the references and in the brief Notes and Remarks sections at the end of each chapter. You will also see that I've included a few exercises to accompany each chapter. These only scratch the surface, of course. Energetic readers may want to seek out greater challenges through the readings suggested in the Notes and Remarks.

My goal was a quick survey of what I perceive to be the major topics in classical Banach space theory: Basis theory, L_p spaces, $C(K)$ spaces, and a brief introduction to the geometry of Banach spaces. But the emphasis here is on *classical*; most of this material is more than thirty years old and, indeed, a great deal of it is more than fifty years old. Readers interested in contemporary research topics in Banach space theory are sure to be disappointed with this modest introduction and are encouraged to look elsewhere.

Finally, I should point out that the course has proven to be of interest to more students than I had originally imagined. Basis theory, for example, has enjoyed a resurgence in certain modern arenas, and such chestnuts as the so-called gliding hump argument frequently resurface in a variety of contemporary research venues. From this point of view, the course has much to offer students interested in operator theory, frames and wavelets, and even in certain corners of algebra such as lattice theory. More important, at least from my point of view, is that the early history of Banach space theory is loaded with elegant, insightful arguments and clever techniques that are not only worthy of study in their own right but are also deserving of greater publicity. It is in this spirit that I offer these notes.

Neal Carothers
Bowling Green, Ohio
February 2003

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Chapter 1

Classical Banach Spaces

To begin, recall that a Banach space is a complete normed linear space. That is, a Banach space is a normed vector space $(X, \|\cdot\|)$ that is a complete metric space under the induced metric $d(x, y) = \|x - y\|$. Unless otherwise specified, we'll assume that all vector spaces are over $\mathbb{R}$, although, from time to time, we will have occasion to consider vector spaces over $\mathbb{C}$.

What follows is a list of the *classical* Banach spaces. Roughly translated, this means the spaces known to Banach. Once we have these examples out in the open, we'll have plenty of time to fill in any unexplained terminology. For now, just let the words wash over you.

The Sequence Spaces ℓ_p and c_0

Arguably the first infinite-dimensional Banach spaces to be studied were the sequence spaces ℓ_p and c_0. To consolidate notation, we first define the vector space s of all real sequences $x = (x_n)$ and then define various subspaces of s.

For each $1 \le p < \infty$, we define

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

and take ℓ_p to be the collection of those $x \in s$ for which $\|x\|_p < \infty$. The inequalities of Hölder and Minkowski show that ℓ_p is a normed space; from there it's not hard to see that ℓ_p is actually a Banach space.

The space ℓ_p is defined in exactly the same way for $0 < p < 1$ but, in this case, $\|\cdot\|_p$ defines a complete quasi-norm. That is, the triangle inequality now holds with an extra constant; specifically, $\|x + y\|_p \le 2^{1/p}(\|x\|_p + \|y\|_y)$. It's worth noting that $d(x, y) = \|x - y\|_p^p$ defines a complete, translation-invariant metric on ℓ_p for $0 < p < 1$.

For $p = \infty$, we define ℓ_∞ to be the collection of all bounded sequences; that is, ℓ_∞ consists of those $x \in s$ for which

$$\|x\|_\infty = \sup_n |x_n| < \infty.$$

It's easy to see that convergence in ℓ_∞ is the same as uniform convergence on $\mathbb{N}$ and, hence, that ℓ_∞ is complete. There are two very natural (closed) subspaces of ℓ_∞: The space c, consisting of all convergent sequences, and the space c_0, consisting of all sequences converging to 0. It's not hard to see that c and c_0 are also Banach spaces.

As subsets of s we have

$$\ell_1 \subset \ell_p \subset \ell_q \subset c_0 \subset c \subset \ell_\infty \tag{1.1}$$

for any $1 < p < q < \infty$. Moreover, each of the inclusions is norm one:

$$\|x\|_1 \geq \|x\|_p \geq \|x\|_q \geq \|x\|_\infty. \tag{1.2}$$

It's of some interest here to point out that, although s is not itself a normed space, it is, at least, a complete metric space under the so-called Fréchet metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. \tag{1.3}$$

Clearly, convergence in the Fréchet metric implies coordinatewise convergence.

Finite-Dimensional Spaces

We will also have occasion to consider the finite-dimensional versions of the ℓ_p spaces. We write ℓ_p^n to denote $\mathbb{R}^n$ under the ℓ_p norm. That is, ℓ_p^n is the space of all sequences $x = (x_1, \ldots, x_n)$ of length n and is supplied with the norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $p < \infty$, and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

for $p = \infty$.

Recall that all norms on $\mathbb{R}^n$ are equivalent. In particular, given any norm $\|\cdot\|$ on $\mathbb{R}^n$, we can find a positive, finite constant C such that

$$C^{-1}\|x\|_1 \leq \|x\| \leq C\|x\|_1 \tag{1.4}$$

for all $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. Thus, convergence in any norm on $\mathbb{R}^n$ is the same as "coordinatewise" convergence and, hence, every norm on $\mathbb{R}^n$ is complete.

Because every finite-dimensional normed space is just "$\mathbb{R}^n$ in disguise," it follows that every finite-dimensional normed space is complete.

The L_p Spaces

We first define the vector space $L_0[0,1]$ to be the collection of all (equivalence classes, under equality almost everywhere [a.e.], of) Lebesgue-measurable functions $f : [0,1] \to \mathbb{R}$. For our purposes, L_0 will serve as the "measurable analogue" of the sequence space s.

For $1 \le p < \infty$, the Banach space $L_p[0,1]$ consists of those $f \in L_0[0,1]$ for which

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} < \infty.$$

The space $L_\infty[0,1]$ consists of all (essentially) bounded $f \in L_0[0,1]$ under the essential supremum norm

$$\|f\|_\infty = \operatorname*{ess.sup}_{0 \le x \le 1} |f(x)| = \inf\{B : |f| \le B \text{ a.e.}\}$$

(in practice, though, we often just write "sup" in place of "ess.sup"). Again, the inequalities of Hölder and Minkowski play an important role here.

As before, the spaces $L_p[0,1]$ are also defined for $0 < p < 1$, but $\|\cdot\|_p$ defines only a quasi-norm. Again, $d(f,g) = \|f - g\|_p^p$ defines a complete, translation-invariant metric on L_p for $0 < p < 1$. The space $L_0[0,1]$ is given the topology of convergence (locally) in measure. For Lebesgue measure on $[0,1]$, this topology is known to be equivalent to that given by the metric

$$d(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx. \tag{1.5}$$

As subsets of $L_0[0,1]$, we have the following inclusions:

$$L_1[0,1] \supset L_p[0,1] \supset L_q[0,1] \supset L_\infty[0,1], \tag{1.6}$$

for any $1 < p < q < \infty$. Moreover, the inclusion maps are all norm one:

$$\|f\|_1 \le \|f\|_p \le \|f\|_q \le \|f\|_\infty. \tag{1.7}$$

The spaces $L_p(\mathbb{R})$ are defined in much the same way but satisfy *no* such inclusion relations. That is, for any $p \ne q$, we have $L_p(\mathbb{R}) \not\subset L_q(\mathbb{R})$. Nevertheless, you may find it curious to learn that $L_p(\mathbb{R})$ and $L_p[0,1]$ are linearly *isometric*.

More generally, given a measure space (X, Σ, μ), we might consider the space $L_p(\mu)$ consisting of all (equivalence classes of) Σ-measurable functions $f : X \to \mathbb{R}$ under the norm

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p}$$

(with the obvious modification for $p = \infty$).

It is convenient to consider at least one special case here: Given any set Γ, we define $\ell_p(\Gamma) = L_p(\Gamma, 2^\Gamma, \mu)$, where μ is counting measure on Γ. What this means is that we identify functions $f : \Gamma \to \mathbb{R}$ with "sequences" $x = (x_\gamma)$ in the usual way: $x_\gamma = f(\gamma)$, and we define

$$\|x\|_p = \left(\sum_{\gamma \in \Gamma} |x_\gamma|^p \right)^{1/p} = \left(\int_\Gamma |f(\gamma)|^p \, d\mu(\gamma) \right)^{1/p} = \|f\|_p$$

for $p < \infty$. Please note that if $x \in \ell_p(\Gamma)$, then $x_\gamma = 0$ for all but countably many γ. For $p = \infty$, we set

$$\|x\|_\infty = \sup_{\gamma \in \Gamma} |x_\gamma| = \sup_{\gamma \in \Gamma} |f(\gamma)| = \|f\|_\infty.$$

We also define $c_0(\Gamma)$ to be the space of all those $x \in \ell_\infty(\Gamma)$ for which the set $\{\gamma : |x_\gamma| > \varepsilon\}$ is *finite* for any $\varepsilon > 0$. Again, this forces an element of $c_0(\Gamma)$ to have countable support. Clearly, $\ell_p(\mathbb{N}) = \ell_p$ and $c_0(\mathbb{N}) = c_0$.

A priori, the Banach space characteristics of $L_p(\mu)$ will depend on the underlying measure space (X, Σ, μ). As it happens, though, Lebesgue measure on $[0, 1]$ and counting measure on $\mathbb{N}$ are essentially the only two cases we have to worry about. It follows from a deep result in abstract measure theory (Maharam's theorem [97]) that every complete measure space can be decomposed into "nonatomic" parts (copies of $[0, 1]$) and "purely atomic" parts (counting measure on some discrete space). From a Banach space point of view, this means that every L_p space can be written as a direct sum of copies of $L_p[0, 1]$ and $\ell_p(\Gamma)$ (or ℓ_p^n).

For the most part we will divide our efforts here into three avenues of attack: Those properties of L_p spaces that don't depend on the underlying measure space, those that are peculiar to $L_p[0, 1]$, and those that are peculiar to the ℓ_p spaces.

The $C(K)$ Spaces

Perhaps the earliest known example of a Banach space is the space $C[a, b]$ of all continuous real-valued functions $f : [a, b] \to \mathbb{R}$ supplied with the "uniform norm":

$$\|f\| = \max_{a \le t \le b} |f(t)|.$$

More generally, if K is any compact Hausdorff space, we write $C(K)$ to denote the Banach space of all continuous real-valued functions $f : K \to \mathbb{R}$ under the norm

$$\|f\| = \max_{t \in K} |f(t)|.$$

For obvious reasons, we sometimes write the norm in $C(K)$ as $\|f\|_\infty$ and refer to it as the "sup norm." In any case, convergence in $C(K)$ is the same as uniform convergence on K.

In Banach's day, point set topology was still very much in its developmental stages. In his book [6], Banach considered $C(K)$ spaces only in the case of compact *metric* spaces K. We, on the other hand, may have occasion to venture further. At the very least, we will consider the case in which K is a compact Hausdorff space (since the theory is nearly identical in this case). And, if we really get ambitious, we may delve into more esoteric settings. For the sake of future reference, here is a brief summary of the situation.

If X is any topological space, we write $C(X)$ to denote the algebra of all real-valued continuous functions $f : X \to \mathbb{R}$. For general X, though, $C(X)$ may not be metrizable. If X is Hausdorff and σ-compact, say $X = \bigcup_{n=1}^{\infty} K_n$, then $C(X)$ is a complete metric space under the topology of "uniform convergence on compacta" (or the "compact-open" topology). This topology is generated by the so-called Fréchet metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}, \tag{1.8}$$

where $\|f\|_n$ is the norm of $f|_{K_n}$ in $C(K_n)$.

If we restrict our attention to the *bounded* functions in $C(X)$, then we may at least apply the sup norm; for this reason, we consider instead the Banach space $C_b(X)$ of all bounded, continuous, real-valued functions $f : X \to \mathbb{R}$ endowed with the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Obviously, $C_b(X)$ is a closed subspace of $\ell_\infty(X)$. If X is at least completely regular, then $C_b(X)$ contains as much information as $C(X)$ itself in the sense that the topology on X is completely determined by knowing the bounded, continuous, real-valued functions on X.

If X is noncompact, then we might also consider the normed space $C_C(X)$ of all continuous $f : X \to \mathbb{R}$ with *compact support*. That is, $f \in C_C(X)$ if f is continuous and if the *support* of f, namely, the set

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}},$$

is compact. Although we may apply the sup norm to $C_C(X)$, it's not, in general, complete. The completion of $C_C(X)$ is the space $C_0(X)$ consisting of all those continuous $f : X \to \mathbb{R}$ that "vanish at infinity." Specifically, $f \in C_0(X)$ if f is continuous and if, for each $\varepsilon > 0$, the set $\{|f| \geq \varepsilon\}$ has compact closure. The space $C_0(X)$ is a closed subspace of $C_b(X)$ and hence is a Banach space under the sup norm.

If X is compact, then, of course, $C_C(X) = C_b(X) = C(X)$. For general X, however, the best we can say is

$$C_C(X) \subset C_0(X) \subset C_b(X) \subset C(X).$$

At least one easy example might be enlightening here: Consider the case $X = \mathbb{N}$; obviously, $\mathbb{N}$ is locally compact and metrizable. Now *every* function $f : \mathbb{N} \to \mathbb{R}$ is continuous, and any such function can quite plainly be identified with a sequence; namely, its range $(f(n))$. That is, we can identify $C(\mathbb{N})$ with s by way of the correspondence $f \in C(\mathbb{N}) \longleftrightarrow x \in s$, where $x_n = f(n)$. Convince yourself that

$$C_b(\mathbb{N}) = \ell_\infty, \quad C_0(\mathbb{N}) = c_0, \quad C_0(\mathbb{N}) \oplus \mathbb{R} = c, \tag{1.9}$$

and that

$$C_C(\mathbb{N}) = \{x \in s : x_n = 0 \text{ for all but finitely many } n\}. \tag{1.10}$$

While this is curious, it doesn't quite tell the whole story. Indeed, both ℓ_∞ and c are actually $C(K)$ spaces. To get a glimpse into why this is true, consider the space $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, the *one-point compactification* of $\mathbb{N}$ (that is, we append a "point at infinity"). If we define a neighborhood of ∞ to be any set with finite (compact) complement, then $\mathbb{N}^*$ becomes a compact Hausdorff space. Convince yourself that

$$c = C(\mathbb{N}^*) \quad \text{and} \quad c_0 = \{f \in C(\mathbb{N}^*) : f(\infty) = 0)\}. \tag{1.11}$$

We'll have more to say about these ideas later.

Hilbert Space

As you'll no doubt recall, the spaces ℓ_2 and L_2 are both Hilbert spaces, or complete inner product spaces. Recall that a vector space H is called a Hilbert space if H is endowed with an inner product $\langle \cdot, \cdot \rangle$ with the property that the induced norm, defined by

$$\|x\| = \sqrt{\langle x, x \rangle}, \tag{1.12}$$

is complete. It is most important here to recognize that the norm in H is intimately related to an inner product by way of (1.12). This is a tall order for the run-of-the-mill norm. From this point of view, Hilbert spaces are quite rare among the teeming masses of Banach spaces.

There is a critical distinction to be made here; perhaps an example will help to explain. Let X denote the space ℓ_2 supplied with the norm $\|x\| = \|x\|_2 + \|x\|_\infty$. Then X is isomorphic (linearly homeomorphic) to ℓ_2 because our new norm satisfies $\|x\|_2 \le \|x\| \le 2\|x\|_2$. But X is *not* itself a Hilbert space. The test is whether the *parallelogram law* holds:

$$\|x + y\|^2 + \|x - y\|^2 \stackrel{?}{=} 2\left(\|x\|^2 + \|y\|^2\right).$$

And it's easy to check that the parallelogram law fails if $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, \ldots)$, for instance. The moral here is that it's not enough to have a well-defined inner product, nor is it enough to have a norm that is close to a known Hilbert space norm. In a Hilbert space, the norm and the inner product are inextricably bound together through equation (1.12).

Hilbert spaces exhibit another property that is rare among the Banach spaces: In a Hilbert space, every closed subspace is the range of a continuous projection. This is far from the case in a general Banach space. (In fact, it is known that any space with this property is already isomorphic to Hilbert space.)

"Neoclassical" Spaces

We have more or less exhausted the list of spaces that were well known in Banach's time. But we have by no means even begun to list the spaces that are commonplace these days. In fact, it would take pages and pages of definitions to do so. For now we'll content ourselves with the understanding that all of the known examples are, in a sense, generalizations of the spaces we have seen thus far.

The Big Questions

We're typically interested in both the *isometric* as well as the *isomorphic* character of a Banach space. (For our purposes, all isometries are *linear*.