WILEY FINANCE

Mathematics and Statistics for Financial Risk Management

Michael B. Miller

Mathematics and Statistics for Financial Risk Management

MICHAEL B. MILLER

WILEY

John Wiley & Sons, Inc.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Miller, Michael B.
Mathematics and statistics for financial risk management / Michael B. Miller.
p. cm. – (Wiley finance)
Includes bibliographical references and index.
ISBN 978-1-118-17062-5 (hardback); ISBN 978-1-118-22777-0 (ebk);
ISBN 978-1-118-24419-7 (ebk); ISBN 978-1-118-23976-6 (ebk)
1. Risk management–Mathematical models. 2. Risk management–Statistical methods.
I. Title.
HD61.M537 2012
332.01'5195–dc23

2011039256

ISBN 978-1-118-17062-5

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

Mathematics and Statistics for Financial Risk Management

Founded in 1807, John Wiley & Sons is the oldest independent publishing company in the United States. With offices in North America, Europe, Australia, and Asia, Wiley is globally committed to developing and marketing print and electronic products and services for our customers' professional and personal knowledge and understanding.

The Wiley Finance series contains books written specifically for finance and investment professionals as well as sophisticated individual investors and their financial advisors. Book topics range from portfolio management to e-commerce, risk management, financial engineering, valuation, and financial instrument analysis, as well as much more.

For a list of available titles, visit our Web site at www.WileyFinance.com.

Preface

The recent financial crisis and its impact on the broader economy underscore the importance of financial risk management in today's world. At the same time, financial products and investment strategies are becoming increasingly complex. It is more important than ever that risk managers possess a sound understanding of mathematics and statistics.

Mathematics and Statistics for Financial Risk Management is a guide to modern financial risk management for both practitioners and academics. Risk management has made great strides in recent years. Many of the mathematical and statistical tools used in risk management today were originally adapted from other fields. As the field has matured, risk managers have refined these tools and developed their own vocabulary for characterizing risk. As the field continues to mature, these tools and vocabulary are becoming increasingly standardized. By focusing on the application of mathematics and statistics to actual risk management problems, this book helps bridge the gap between mathematics and statistics in theory and risk management in practice.

Each chapter in this book introduces a different topic in mathematics or statistics. As different techniques are introduced, sample problems and application sections demonstrate how these techniques can be applied to actual risk management problems. Exercises at the end of each chapter, and the accompanying solutions at the end of the book, allow readers to practice the techniques they are learning and to monitor their progress.

This book assumes that readers have a solid grasp of algebra and at least a basic understanding of calculus. Even though most chapters start out at a very basic level, the pace is necessarily fast. For those who are already familiar with the topic, the beginning of each chapter will serve as a quick review and as an introduction to certain vocabulary terms and conventions. Readers who are new to these topics may find they need to spend more time in the initial sections.

Risk management in practice often requires building models using spreadsheets or other financial software. Many of the topics in this book are accompanied by a icon, as shown here. These icons indicate that Excel examples can be found at John Wiley & Sons' companion website for

Mathematics and Statistics for Financial Risk Management, at www.wiley.com/go/millerfinance.

You can also visit the author's web site, www.risk256.com, for the latest financial risk management articles, code samples, and more. To provide feedback, you can contact the author at mike@risk256.com.

Acknowledgments

Like most of today's risk managers, I learned much of what I know about risk management on the job. I was fortunate to work with some very knowledgeable individuals early in my career. In particular, I would like to thank Gideon Pell and Kent Osband.

This book would not have been possible without the help of many individuals. I would like to thank Jeffrey Garnett, Steve Lerit, Hyunsuk Moon, Elliot Noma, and Eldar Radovici for taking time to read early drafts. The book is certainly better for their comments and feedback. I would also like to thank everybody at John Wiley & Sons for their help in bringing this book together.

Finally, and most importantly, I would like to thank my wife, Amy, who not only read over early drafts and talked me through a number of decisions, but also put up with countless nights and weekends of typing and editing. For this and much, much more, thank you.

Contents

CHAPTER 1

Some Basic Math

In this chapter we will review three math topics—logarithms, combinatorics, and geometric series—and one financial topic, discount factors. Emphasis will be given to the specific aspects of these topics that are most relevant to risk management.

LOGARITHMS

In mathematics, logarithms, or logs, are related to exponents, as follows:

$$\log_b a = x \Leftrightarrow a = b^x \tag{1.1}$$

We say, "The log of a, base b, equals x, which implies that a equals b to the x and vice versa." If we take the log of the right-hand side of Equation 1.1 and use the identity from the left-hand side of the equation, we can show that:

$$\log_b(b^x) = x \tag{1.2}$$

Taking the log of b^x effectively cancels out the exponentiation, leaving us with x.

An important property of logarithms is that the logarithm of the product of two variables is equal to the sum of the logarithms of those two variables. For two variables, X and Y:

$$\log_b(XY) = \log_b X + \log_b Y \tag{1.3}$$

Similarly, the logarithm of the ratio of two variables is equal to the difference of their logarithms:

$$\log_b\left(\frac{X}{Y}\right) = \log_b X - \log_b Y \tag{1.4}$$

If we replace Y with X in Equation 1.3, we get:

$$\log_b(X^2) = 2\log_b X \tag{1.5}$$

We can generalize this result to get the following power rule:

$$\log_b(X^n) = n\log_b X \tag{1.6}$$

In general, the base of the logarithm, b, can have any value. Base 10 and base 2 are popular bases in certain fields, but in many fields, and especially in finance, e, Euler's number, is by far the most popular. Base e is so popular that mathematicians have given it its own name and notation. When the base of a logarithm is e, we refer to it as a *natural logarithm*. In formulas, we write:

$$\ln(a) = x \Leftrightarrow a = e^x \tag{1.7}$$

From this point on, unless noted otherwise, assume that any mention of logarithms refers to natural logarithms.

Logarithms are defined for all real numbers greater than or equal to zero. Figure 1.1 shows a plot of the logarithm function. The logarithm of zero is negative infinity, and the logarithm of one is zero. The function

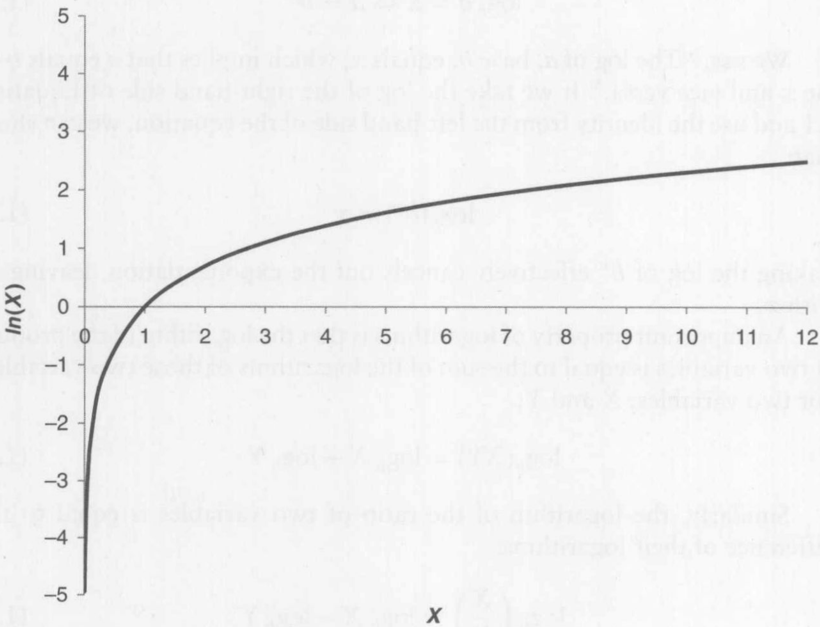

FIGURE 1.1 Natural Logarithm

grows without bound; that is, as X approaches infinity, the $\ln(X)$ approaches infinity as well.

LOG RETURNS

One of the most common applications of logarithms in finance is computing log returns. Log returns are defined as follows:

$$r_t \equiv \ln(1 + R_t) \quad \text{where} \quad R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1.8}$$

Here r_t is the log return at time t, R_t is the standard or simple return, and P_t is the price of the security at time t. We use this convention of capital R for simple returns and lowercase r for log returns throughout the rest of the book. This convention is popular, but by no means universal. Also, be careful: Despite the name, the log return is not the log of R_t, but the log of $(1 + R_t)$.

For small values, log returns and simple returns will be very close in size. A simple return of 0% translates exactly to a log return of 0%. A simple return of 10% translates to a log return of 9.53%. That the values are so close is convenient for checking data and preventing operational errors. Table 1.1 shows some additional simple returns along with their corresponding log returns.

TABLE 1.1 Log Returns and Simple Returns

R	$\ln(1 + R)$
1.00%	1.00%
5.00%	4.88%
10.00%	9.53%
20.00%	18.23%

To get a more precise estimate of the relationship between standard returns and log returns, we can use the following approximation:*

$$r \approx R - \frac{1}{2}R^2 \tag{1.9}$$

*This approximation can be derived by taking the Taylor expansion of Equation 1.8 around zero. Though we have not yet covered the topic, for the interested reader a brief review of Taylor expansions can be found in Appendix B.

As long as R is small, the second term on the right-hand side of Equation 1.9 will be negligible, and the log return and the simple return will have very similar values.

COMPOUNDING

Log returns might seem more complex than simple returns, but they have a number of advantages over simple returns in financial applications. One of the most useful features of log returns has to do with compounding returns. To get the return of a security for two periods using simple returns, we have to do something that is not very intuitive, namely adding one to each of the returns, multiplying, and then subtracting one:

$$R_{2,t} = \frac{P_t - P_{t-2}}{P_{t-2}} = (1 + R_{1,t})(1 + R_{1,t-1}) - 1 \tag{1.10}$$

Here the first subscript on R denotes the length of the return, and the second subscript is the traditional time subscript. With log returns, calculating multiperiod returns is much simpler; we simply add:

$$r_{2,t} = r_{1,t} + r_{1,t-1} \tag{1.11}$$

By substituting Equation 1.8 into Equation 1.10 and Equation 1.11, you can see that these are equivalent. It is also fairly straightforward to generalize this notation to any return length.

SAMPLE PROBLEM

Question:

Using Equation 1.8 and Equation 1.10, generalize Equation 1.11 to returns of any length.

Answer:

$$R_{n,t} = \frac{P_t - P_{t-n}}{P_{t-n}} = \frac{P_t}{P_{t-n}} - 1 = \frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}} \cdots \frac{P_{t-n+1}}{P_{t-n}} - 1$$

$$R_{n,t} = (1 + R_{1,t})(1 + R_{1,t-1}) \cdots (1 + R_{1,t-n+1}) - 1$$

$$(1 + R_{n,t}) = (1 + R_{1,t})(1 + R_{1,t-1}) \cdots (1 + R_{1,t-n+1})$$

$$r_{n,t} = r_{1,t} + r_{1,t-1} + \cdots + r_{1,t-n+1}$$

Note that to get to the last line, we took the logs of both sides of the previous equation, using the fact that the log of the product of any two variables is equal to the sum of their logs, as shown in Equation 1.3.

LIMITED LIABILITY

Another useful feature of log returns relates to limited liability. For many financial assets, including equities and bonds, the most that you can lose is the amount that you've put into them. For example, if you purchase a share of XYZ Corporation for \$100, the most you can lose is that \$100. This is known as limited liability. Today, limited liability is such a common feature of financial instruments that it is easy to take it for granted, but this was not always the case. Indeed, the widespread adoption of limited liability in the nineteenth century made possible the large publicly traded companies that are so important to our modern economy, and the vast financial markets that accompany them.

That you can lose only your initial investment is equivalent to saying that the minimum possible return on your investment is −100%. At the other end of the spectrum, there is no upper limit to the amount you can make in an investment. The maximum possible return is, in theory, infinite. This range for simple returns, −100% to infinity, translates to a range of negative infinity to positive infinity for log returns.

$$\begin{aligned} R_{\min} &= -100\% \Rightarrow r_{\min} = -\infty \\ R_{\max} &= +\infty \Rightarrow r_{\max} = +\infty \end{aligned} \tag{1.12}$$

As we will see in the following chapters, when it comes to mathematical and computer models in finance, it is often much easier to work with variables that are unbounded, that is variables that can range from negative infinity to positive infinity.

GRAPHING LOG RETURNS

Another useful feature of log returns is how they relate to log prices. By rearranging Equation 1.10 and taking logs, it is easy to see that:

$$r_t = p_t - p_{t-1} \tag{1.13}$$