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Large-Order Behaviour of Perturbation Theory

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Preface

This volume is devoted to the determination of the behaviour of perturbation theory at large orders in quantum mechanics and quantum field theory, and its application to the problem of summation of perturbation series.

Perturbation theory, useful in quantum mechanics for systems with several degrees of freedom, is the only known analytic method to calculate physical quantities in quantum field theory. However, following arguments given by Dyson, it has long been suspected that the perturbation series in field theory was divergent. Many years have been necessary to develop tools to deal with this question. In field theory, the representation of Green's (or correlation) functions in terms of functional integrals has played an essential role. It has then been possible to characterize the behaviour of perturbation theory at large orders for many models: Perturbation series in many quantum-mechanics models and in quantum field theory are indeed only asymptotic and, thus, diverge for all values of the expansion parameter.

The behaviour at large orders provides information about the series domain of validity, whether it defines the theory uniquely (the problem of Borel summability) and suggests methods to extract numerical information from the series when the expansion parameter is not small.

The reprinted articles can be roughly divided in three sets: A first set deals with simple systems of the form of the quartic anharmonic oscillator or the ϕ^4 field theory. It is shown, first in quantum mechanics, how large-order behaviour is related to barrier-penetration effects for unphysical values of the expansion parameter. These effects can be calculated by semiclassical methods of the WKB type as first discussed by Bender and Wu. However, they can also be inferred from a steepest descent calculation of the imaginary time or Euclidean path-integral representation. Barrier penetration is then dominated by finite action solutions of Euclidean classical equations of motions, instantons. The integration over path configurations close to the classical paths leads to some subtle zero mode problems which can be solved by the method of collective coordinates.

Only the instanton method can be easily generalized to quantum field theory, as first shown by Lipatov. New problems arise connected with the need for renormalization. Renormalizable and super-renormalizable methods behave somewhat differently in this respect.

The second set of articles deals with more complex systems in quantum mechanics and general boson field theories, including gauge theories (Abelian and non-Abelian) coupled to scalar fields. From the point of view of Borel summability models fall into two classes: Models in which the classical minimum of the potential is unique, and which has a change to be Borel summable; and models in which the perturbative expansion has been performed around a relative minimum of the potential or in which the minimum of the potential is degenerate and instantons connect the minima, in which the series cannot be Borel summable. In the latter example, which is more physically interesting, the large-order behaviour is not directly connected to a solution of the Euclidean equation of motion but to configurations of pairs of largely separated instantons.

Two-instanton configurations are the simplest example of so-called multi-instanton configurations. Their role has been fully analyzed only in simple quantum-mechanical models. A systematic discussion of this problem in field theory is still lacking. Models containing fermion fields are then discussed. It is shown how the Pauli principle (reflected by the sign of the fermion loops) affect the large-order behaviour.

A third set of articles deals with some problems which arise in the large-order analysis when field theories are just renormalizable. Then contributions to the large-order behaviour come from the large momentum singularities of Feynman diagrams (renormalons). They are related to the first coefficient of the Callan–Symanzik β-function. In the case of infrared-stable (in the renormalization group sense) theories they present Borel summability.

Finally, an application of the large-order behaviour analysis is included: The calculations of critical exponents from ϕ^4 field theory and renormalization group arguments since it relies on a summation of perturbation series based directly on the knowledge of the large-order behaviour.

For completeness some articles have been added which consider the large-order behaviour in the case of non-polynomial potentials, or for the large N expansions in models with $O(N)$ symmetries.

As a conclusion, the large-order behaviour analysis has significantly improved our understanding of perturbation theory, in particular in the field theory context. It has confirmed that perturbation series in field theory are always divergent, but also suggested in some useful cases methods to deal with this difficulty. In particular, it has made reliable calculations of physical quantities based on perturbation theory possible even for not small coupling constants. The evaluation of critical exponents near second-order phase transitions provides a very interesting example.

J.-C. Le Guillou and J. Zinn-Justin

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From Physical Review Letters:
C.M. Bender and T.T. Wu, Statistical analysis of Feynman diagrams, Phys. Rev. Lett. 37 (1976) 117–120; J.E. Avron, B.G. Adams, J. Čížek, M. Clay, M.L. Glasser, P. Otto, J. Paldus and E. Vrscay, Bender–Wu formula, the SO(4,2) dynamical group, and the Zeeman effect in hydrogen, Phys. Rev. Lett. 43 (1979) 691–693; I.W. Herbst and B. Simon, Stark effect revisited, Phys. Rev. Lett. 41 (1978) 67–69; L. Benassi, V. Grecchi, E. Harrell and B. Simon, Bender–Wu formula and the Stark effect in hydrogen, Phys. Rev. Lett. 42 (1979) 704–707 and 1430 (erratum); R.J. Damburg, R.Kh. Propin, S. Graffi, V. Grecchi, E.M. Harrell II, J. Čížek, J. Paldus and H.J. Silverstone, $1/R$ expansion for H_2^+: analyticity, summability, asymptotics, and calculation of exponentially small terms, Phys. Rev. Lett. 52 (1984) 1112–1115; C. Itzykson, G. Parisi and J.B. Zuber, Asymptotic estimates in scalar electrodynamics, Phys. Rev. Lett. 38 (1977) 306–310; T. Kinoshita and W.B. Lindquist, Eighth-order anomalous magnetic moment

of the electron, Phys. Rev. Lett. 47 (1981) 1573–1576; G.A. Baker, Jr., B.G. Nickel, M.S. Green and D.I. Meiron, Ising-model critical indices in three dimensions from the Callan–Symanzik equation, Phys. Rev. Lett. 36 (1976) 1351–1354; J.C. Le Guillou and J. Zinn-Justin, Critical exponents for the n-vector model in three dimensions from field theory, Phys. Rev. Lett. 39 (1977) 95–98.
From Soviet Physics JETP:
L.N. Lipatov, Divergence of the perturbation-theory series and the quasi-classical theory, Sov. Phys. JETP 45 (1977) 216–223; A.P. Bukhvostov and L.N. Lipatov, Asymptotic estimates of high order perturbation theory approximations in scalar electrodynamics, Sov. Phys. JETP 46 (1977) 871–879.
From JETP Letters:
L.N. Lipatov, Divergence of the perturbation-theory series and pseudoparticles, JETP Lett. 25 (1977) 104–107.
From Soviet Journal of Nuclear Physics:
E.B. Bogomol'nyĭ and Yu.A. Kubyshin, Asymptotic estimates for graphs with a fixed number of fermion loops in quantum electrodynamics. The choice of the form of the steepest-descent solutions, Sov. J. Nucl. Phys. 34 (1981) 853–858; E.B. Bogomol'nyĭ and Yu.A. Kubyshin, Asymptotic estimates for diagrams with a fixed number of fermion loops in quantum electrodynamics. The extremal configurations with the symmetry group $O(2) \otimes O(3)$, Sov. J. Nucl. Phys. 35 (1982) 114–119.
From Journal of Mathematical Physics:
J. Zinn-Justin, Expansion around instantons in quantum mechanics, J. Math. Phys. 22 (1981) 511–520; J. Zinn-Justin, Instantons in quantum mechanics: Numerical evidence for a conjecture, J. Math. Phys. 25 (1984) 549–555.

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From Journal de Physique Lettres:
E. Brezin and J. Zinn-Justin, Expansion of the H_2^- ground state energy in inverse powers of the distance between the two protons, J. Physique Lett. 40 (1979) L511–L512; J.C. Le Guillou and J. Zinn-Justin, Accurate critical exponents from the ϵ-expansion, J. Physique Lett. 46 (1985) L137–L141.

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Contents

1. Methods and Simple Examples

1.1. Introduction

In the text which follows, we want to present the general ideas basic to the analysis of the behaviour of perturbation theory at large orders in quantum mechanics and in quantum-field theory, and introduce the articles which are reprinted in this volume. Additional references to articles which, for lack of space, could not be included in this volume are given in section 5. In the text, they are referred by [S...] to distinguish them from references to the reprinted articles.

1.2. Large-order behaviour: The intuitive idea

In his 1951 article about divergence of perturbation theory in QED, Dyson [1] presented the following argument: let us consider a system of N charged particles (N being large) of same charge e. The energy E of the system has the following structure,

$$E \sim NT + \tfrac{1}{2}e^2VN^2, \tag{1.1}$$

in which T is the mean kinetic energy and V characterizes the mean Coulomb potential, the factor $\frac{1}{2}N^2$ counting the number of interacting pairs. For e^2 positive, the ground state of such a system is stable. However, if e^2 is negative, i.e., if the Coulomb potential between identical charges becomes attractive, then the ground-state energy first increases with N for N small, but when N becomes larger than some critical value N_c,

$$N_c = -T/Ve^2, \tag{1.2}$$

the ground-state energy starts decreasing and behaves like $-N^2$ for N large. In a relativistic system, pair creation is possible. Therefore, even if we begin with a state containing no charged particles, due to quantum-barrier penetration effects an infinite number of pairs will be created, and a state of infinite negative energy generated. For all values $e^2 < 0$, the Hamiltonian cannot be bounded from below. We first conclude that in QED physical quantities cannot be analytic functions of e^2, they must have a singularity at $e^2 = 0$. Furthermore, the perturbation series will start being sensitive to this effect at order N_c when the diagrams corresponding to the creation of N_c charged particles appear. One, therefore, expects that the terms in the perturbation series will decrease for an order k, $k < N_c$, and increase at higher orders. Let us write the expansion of some physical quantity $F(e^2)$ as

$$F(e^2) = \sum_{k=0}^{\infty} F_k (e^2)^k. \tag{1.3}$$

We infer that

$$F_{k-1}/F_k \sim \frac{1}{e^2} \quad \text{for } k \sim N_c .$$

Since N_c is itself of order $1/e^2$ we find

$$F_{k+1}/F_k \sim k,$$

i.e.,

$$F_k \sim k!. \tag{1.4}$$

We conclude that the ground-state instability for unphysical values of the coupling constants, due to quantum mechanical barrier penetration effects, leads to the divergence of perturbation theory. Moreover, by a calculation of the barrier-penetration coefficient we should be able to estimate more accurately the large-order behaviour.

In the next section, we shall briefly examine the consequences of such a result for perturbation theory. Let us, however, note already that eq. (1.2) does not take into account the Pauli principle with forbids to put an arbitrary number of fermions in the same state. The estimate is really only valid for charged bosons.

1.3. Divergent series: A few remarks

Let $f(z)$ be a function analytic in the domain D,

$$D: \qquad |\arg z| \le \tfrac{1}{2}\alpha, \quad |z| \le \rho . \tag{1.5}$$

We assume that $f(z)$ can be expanded in a power series at the origin,

$$f(z) = \sum_0^{\infty} f_k z^k, \tag{1.6}$$

and that expansion (1.6) is asymptotic to $f(z)$ in D. This means that series (1.6) diverges for $z \neq 0$ and that in D it satisfies the bound

$$\left| f(z) - \sum_{k=0}^{K} f_k z^k \right| \le C_{K+1} |z|^{K+1} \quad \text{for all } K. \tag{1.7}$$

Although the series in eq. (1.6) is always divergent, it can, nevertheless, be used to estimate the function $f(z)$ for $|z|$ small. At $|z|$ fixed, we can look for a minimum of the r.h.s. of expression (1.7) when we vary K. If $|z|$ is small enough, the bound first decreases with K and then, since the series is divergent, finally increases. If we truncate the series at the minimum, we get the best possible estimate of $f(z)$. However, unlike the case of a convergent series, $f(z)$ can be estimated only with a finite accuracy. Let us assume, e.g., that the coefficients C_K have the form

$$C_K = MC^{-K} K!. \tag{1.8}$$

The accuracy is then characterized by a function $\varepsilon(z)$ which is given by

$$\varepsilon(z) = \min_{\{K\}} C_K |z|^K \sim \exp(-C/|z|). \tag{1.9}$$

We see that an asymptotic series does not, in general, define a unique function. If the series is asymptotic to a first function $f(z)$, it is also asymptotic to

$$f(z) + b\exp(-a/z), \quad \text{with} \quad a\cos(\tfrac{1}{2}\alpha) > C,$$

e.g., since this new function is analytic in D and satisfies the bound given in eq. (1.7) in D, at least for $|b|$ small enough. However, there is one situation in which the asymptotic series defines a unique function. If the angle α satisfies the condition

$$\alpha \geq \pi, \tag{1.10}$$

then a classical theorem about analytic functions tells us that a function analytic in D and bounded by $\varepsilon(z)$ in the whole domain vanishes identically.

In the latter case, the next problem is to find a method to reconstruct the function from the series. This question will be examined later.

However, from this discussion we conclude that the problem of the behaviour of perturbation theory at large orders is not only to discover if, as Dyson's simple arguments suggest, the perturbation series is divergent, but, in addition, when it is divergent, whether it determines the physical quantities uniquely or not.

1.4. Large-order behaviour and dispersion relation

Let us assume for simplicity that $f(z)$ is a real function, analytic in the cut plane with a cut along $[-\infty, 0]$, and that it can be represented by a Cauchy integral as

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{0} dz' \frac{\operatorname{Im} f(z')}{z' - z}. \tag{1.11}$$

If $f(z)$ has an asymptotic expansion near $z = 0$,

$$f(z) = \sum_k f_k z^k, \tag{1.12}$$

the coefficients f_k are then given by

$$f_k = \frac{1}{\pi}\int_{-\infty}^{0} \frac{dz}{z^{k+1}} \operatorname{Im} f(z). \tag{1.13}$$

For k large, the Cauchy integral is clearly dominated by the small values of z. Therefore, the large-order behaviour of the coefficients of the asymptotic series can be derived from the knowledge of the behaviour of $\operatorname{Im} f(z)$ for z small and negative [10,11]. Let us assume, e.g.,

$$\operatorname{Im} f(z) \sim z^{-b} e^{a/z}, \tag{1.14}$$