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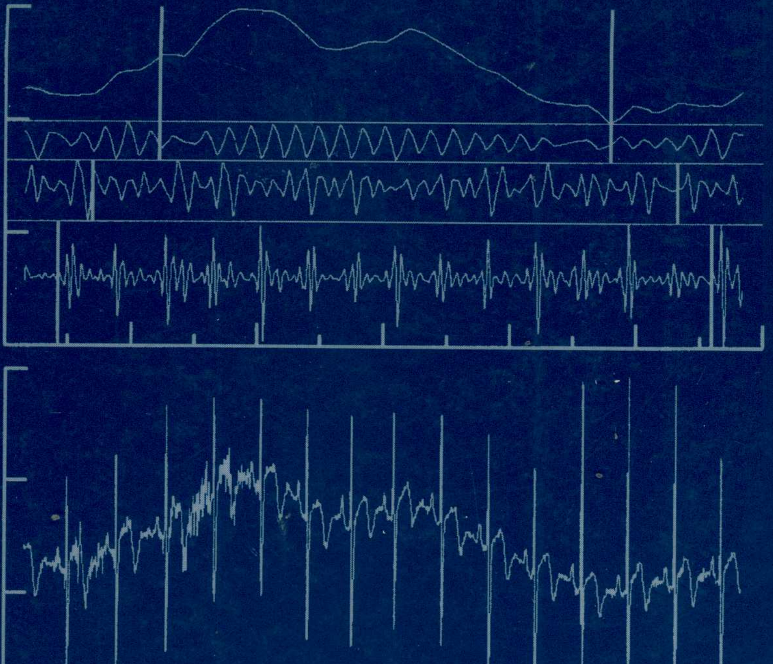

Wavelet Methods for Time Series Analysis

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Preface

The last decade has seen an explosion of interest in wavelets, a subject area that has coalesced from roots in mathematics, physics, electrical engineering and other disciplines. As a result, wavelet methodology has had a significant impact in areas as diverse as differential equations, image processing and statistics. This book is an introduction to wavelets and their application in the analysis of discrete time series typical of those acquired in the physical sciences. While we present a thorough introduction to the basic theory behind the discrete wavelet transform (DWT), our goal is to bridge the gap between theory and practice by

- emphasizing what the DWT actually means in practical terms;
- showing how the DWT can be used to create informative descriptive statistics for time series analysts;
- discussing how stochastic models can be used to assess the statistical properties of quantities computed from the DWT; and
- presenting substantive examples of wavelet analysis of time series representative of those encountered in the physical sciences.

To date, most books on wavelets describe them in terms of continuous functions and often introduce the reader to a plethora of different types of wavelets. We concentrate on developing wavelet methods in discrete time via standard filtering and matrix transformation ideas. We purposely avoid overloading the reader by focusing almost exclusively on the class of wavelet filters described in Daubechies (1992), which are particularly convenient and useful for statistical applications; however, the understanding gained from a study of the Daubechies class of wavelets will put the reader in a excellent position to work with other classes of interest. For pedagogical purposes, this book in fact starts (Chapter 1) and ends (Chapter 11) with discussions of the continuous case. This organization allows us at the beginning to motivate ideas from a historical perspective and then at the end to link ideas arising in the discrete analysis to some of the widely known results for continuous time wavelet analysis.

Topics developed early on in the book (Chapters 4 and 5) include the DWT and the 'maximal overlap' discrete wavelet transform (MODWT), which can be regarded as a generalization of the DWT with certain quite appealing properties. As a whole, these two chapters provide a self-contained introduction to the basic properties of wavelets, with an emphasis both on algorithms for computing the DWT and MODWT and also on the use of these transforms to provide informative descriptive statistics for time series. In particular, both transforms lead to both a scale-based decomposition of the sample variance of a time series and also a scale-based additive decomposition known as a multiresolution analysis. A generalization of the DWT and MODWT that are known in the literature as 'wavelet packet' transforms, and the decomposition of time series via matching pursuit, are among the subjects of Chapter 6. In the second part of the book, we combine these transforms with stochastic models to develop wavelet-based statistical inference for time series analysis. Specific topics covered in this part of the book include

- the wavelet variance, which provides a scale-based analysis of variance complementary to traditional frequency-based spectral analysis (Chapter 8);
- the analysis and synthesis of 'long memory processes,' i.e., processes with slowly decaying correlations (Chapter 9); and
- signal estimation via 'thresholding' and 'denoising' (Chapter 10).

This book is written 'from the ground level and up.' We have attempted to make the book as self-contained as possible (to this end, Chapters 2, 3 and 7 contain reviews of, respectively, relevant Fourier and filtering theory; key ideas in the orthonormal transforms of time series; and important concepts involving random variables and stochastic processes). The text should thus be suitable for advanced undergraduates, but is primarily intended for graduate students and researchers in statistics, electrical engineering, physics, geophysics, astronomy, oceanography and other physical sciences. Readers with a strong mathematical background can skip Chapters 2 and 3 after a quick perusal. Those with prior knowledge of the DWT can make use of the Key Facts and Definitions toward the end of various sections in Chapters 4 and 5 to assess how much of these sections they need to study. This book – or drafts thereof – have been used as a textbook for a graduate course taught at the University of Washington for the past ten years, but we have also designed it to be a self-study work-book by including a large number of exercises embedded within the body of the chapters (particularly Chapters 2 to 5), with solutions provided in the Appendix. Working the embedded exercises will provide readers with a means of progressively understanding the material. For use as a course textbook, we have also provided additional exercises at the end of each chapter (instructors wishing to obtain a solution guide for the exercises should follow the guidance given on the Web site detailed below).

The wavelet analyses of time series that are described in Chapters 4 and 5 can readily be carried out once the basic algorithms for computing the DWT and MODWT (and their inverses) are implemented. While these can be immediately and readily coded up using the pseudo-code in the Comments and Extensions to Sections 4.6 and 5.5, links to existing software in `S-Plus`, `R`, `MATLAB` and Lisp can be found by consulting the Web site for this book, which currently is at

`http://faculty.washington.edu/dbp/wmtsa.html`

The reader should also consult this Web site to obtain a current errata sheet and to download the coefficients for various scaling filters (as discussed in Sections 4.8 and 4.9), the values for all the time series used as examples in this book, and certain computed values that can be used to check computer code. To facilitate preparation of overheads for courses and seminars, the Web site also allows access to pdf files with all the figures and tables in the book (please note that these figures and tables are the copyright of Cambridge University Press and must not be further distributed or used without written permission).

The book was written using Donald Knuth's superb typesetting system TEX as implemented by Blue Sky Research in their product TEXtures for Apple Macintosh™ computers. The figures in this book were created using either the plotting system GPL written by W. Hess (whom we thank for many years of support) or `S-Plus`, the commercial version of the `S` language developed by J. Chambers and co-workers and marketed by MathSoft, Inc. The computations necessary for the various examples and figures were carried out using either `S-Plus` or PITSSA (a Lisp-based object-oriented program for interactive time series and signal analysis that was developed in part by one of us (Percival)).

We thank R. Spindel and the late J. Harlett of the Applied Physics Laboratory, University of Washington, for providing discretionary funding that led to the start of this book. We thank the National Science Foundation, the National Institutes of Health, the Environmental Protection Agency (through the National Research Center for Statistics and the Environment at the University of Washington), the Office of Naval Research and the Air Force Office of Scientific Research for ongoing support during the writing of this book. Our stay at the Isaac Newton Institute for Mathematical Sciences (Cambridge University) during the program on Nonlinear and Nonstationary Signal Processing in 1998 contributed greatly to the completion of this book; we thank the Engineering and Physical Science Research Council (EPSRC) for the support of one of us (Percival) through a Senior Visiting Fellowship while at Cambridge.

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Conventions and Notation

- *Important conventions*

(83)	refers to the single displayed equation on page 83
(69a), (69b)	refers to different displayed equations on page 69
Figure 86	refers to the figure on page 86
Table 109	refers to the table on page 109
Exercise [72]	refers to the embedded exercise on page 72 (an answer is in the Appendix)
Exercise [4.9]	refers to the ninth exercise at the end of Chapter 4
$H(\cdot)$	refers to a function
$H(f)$	refers to the value of the function $H(\cdot)$ at f
$\{h_l\}$	refers to a sequence of values indexed by the integer l
h_l	refers to the lth value of a sequence

In the following lists, the numbers at the end of the brief descriptions are page numbers where more information about – or an example of the use of – an abbreviation or symbol can be found.

- *Abbreviations used frequently*

ACS	autocorrelation sequence	15, 266, 341
ACVS	autocovariance sequence	266
ANOVA	analysis of variance	19, 67
AR	autoregressive process	268
ARFIMA	autoregressive, fractionally integrated, moving average process	285
CWT	continuous wavelet transform	1, 10
dB	decibels, i.e., $10 \log_{10}(\cdot)$	73

DFBM	discrete fractional Brownian motion	279
DFT	discrete Fourier transform	22
DHM	Davies–Harte method	290
DWPT	discrete wavelet packet transform	206, 209
DWT	discrete wavelet transform	1, 13, 56
ECG	electrocardiogram	125
EDOF	equivalent degrees of freedom	313
FBM	fractional Brownian motion	279
FD	fractionally differenced	281
FFT	fast Fourier transform	28
FGN	fractional Gaussian noise	279
GSSM	Gaussian spectral synthesis method	291
Hz	Hertz: 1 Hz = 1 cycle per second	48
IID	independent and identically distributed	262
LA	least asymmetric	107
LSE	least squares estimate or estimator	374, 378
MAD	median absolute deviation	420
MODWPT	maximal overlap discrete wavelet packet transform	207, 231
MODWT	maximal overlap discrete wavelet transform	159
ML	maximum likelihood	341, 361
MLE	maximum likelihood estimate or estimator	341, 361
MRA	multiresolution analysis	65, 461
MRC	mobile radio communications	436
NMR	nuclear magnetic resonance	420
NPES	normalized partial energy sequence	129, 394–5
ODFT	orthonormal discrete Fourier transform	41, 46
OLSE	ordinary least squares estimate or estimator	378
PDF	probability density function	256
PPL	pure power law	281
QMF	quadrature mirror filter	75, 474
RMSE	root mean square error	364, 436
RV	random variable	256
SDF	spectral density function	267
SURE	Stein's unbiased risk estimator	404
WLSE	weighted least squares estimate or estimator	374
WP	wavelet packet	209

- *Non-Greek notation used frequently*

A_j	integral of squared SDF $S_j^2(\cdot)$ for $\{\overline{W}_{j,t}\}$	307
$\mathcal{A}_j$	$N_j \times N_{j-1}$ matrix (rows have $\{g_l\}$ periodized to N_{j-1})	94
$\widetilde{\mathcal{A}}_j$	$N \times N$ matrix (rows have upsampled $\{\tilde{g}_l\}$ periodized to N)	176

$\mathcal{A}_L(\cdot)$	squared gain function for low pass component of $\mathcal{H}^{(\mathrm{D})}(\cdot)$	106
$\{a_{n,t}\}$	nth order sine data taper	274
$\arg(z)$	argument of complex-valued number z	21
B	backward shift operator	283
B_t	discrete fractional Brownian motion (DFBM)	279
$B_H(\cdot)$	fractional Brownian motion (FBM)	279
$\mathcal{B}_j$	$N_j \times N_{j-1}$ matrix (rows have $\{h_l\}$ periodized to N_{j-1})	94
$\widetilde{\mathcal{B}}_j$	$N \times N$ matrix (rows have upsampled $\{\tilde{h}_l\}$ periodized to N)	176
C_j	average value of SDF $S_X(\cdot)$ over octave band $[\frac{1}{2^{j+1}}, \frac{1}{2^j}]$	343
$\widetilde{C}_j$	approximation to C_j for FD processes	344
$\{C_n\}$	normalized partial energy sequence (NPES)	129, 395
$\mathbf{C}$	N dimensional stochastic signal vector	393
$\operatorname{cov}\{\cdot,\cdot\}$	covariance operator	259
$\mathcal{D}(\cdot)$	squared gain function for difference filter	105
$\mathcal{D}_j$	jth level wavelet detail (DWT)	64
$\widetilde{\mathcal{D}}_j$	jth level wavelet detail (MODWT)	169, 171
$\mathbb{D}$	dictionary (collection of vectors) used in matching pursuit	239
$\mathbf{D}$	N dimensional deterministic signal vector	393
d	number of differencing operations	287
$d_{j,t}$	tth component on jth level of $\mathbf{d}$ for DWT	419
d_l	lth component of vector $\mathbf{d}$	398
$\mathbf{d}$	transform coefficients for deterministic signal $\mathbf{D}$	398
$\mathbf{d}_\gamma$	dictionary element (vector in matching pursuit dictionary $\mathbb{D}$)	239
$E\{\cdot\}$	expectation operator	256, 258
$E\{X_0 \mid X_1 = x_1\}$	conditional expectation of X_0 given $X_1 = x_1$	260
$\mathcal{E}_{\mathbf{X}}$	energy (squared norm) for vector $\mathbf{X}$	42, 72
e	$2.718281828459045\cdots$	3, 21
e^{ix}	complex exponential	21
$e_{j,t}$	tth component on jth level of $\mathbf{e}$ for DWT	419
e_l	lth component of vector $\mathbf{e}$	398
$\mathbf{e}$	transform coefficients for IID noise $\boldsymbol{\epsilon}$	398
$\{F_k\}$	orthonormal discrete Fourier transform (ODFT) coefficients	46
$\mathcal{F}$	$N \times N$ orthonormal discrete Fourier transform matrix	47
$\mathbf{F}$	vector containing ODFT coefficients $\{F_k\}$	47
f	frequency of a sinusoid	22
f_k	k/N or $k/(N\,\Delta t)$, the kth Fourier frequency	28, 87
$f_{\mathcal{N}}$	Nyquist frequency	87, 267
$f_X(\cdot)$	probability density function (PDF) for RV X	256
$f_{X_0,X_1}(\cdot,\cdot)$	joint PDF for RVs X_0 and X_1	258
$f_{X_0 \mid X_1 = x_1}(\cdot)$	conditional PDF for RV X_0 given $X_1 = x_1$	260
$G(\cdot)$	transfer function for $\{g_l\}$	76, 154

$\widetilde{G}(\cdot)$	transfer function for $\{\tilde{g}_l\}$	163, 202
$G_j(\cdot)$	transfer function for $\{g_{j,l}\}$, with $G_1(\cdot) \equiv G(\cdot)$	97, 154
$\widetilde{G}_j(\cdot)$	transfer function for $\{\tilde{g}_{j,l}\}$, with $\widetilde{G}_1(\cdot) \equiv \widetilde{G}(\cdot)$	169, 202
$\mathcal{G}(\cdot)$	squared gain function for $\{g_l\}$	76, 154
$\widetilde{\mathcal{G}}(\cdot)$	squared gain function for $\{\tilde{g}_l\}$	163, 202
$\mathcal{G}_j(\cdot)$	squared gain function for $\{g_{j,l}\}$, with $\mathcal{G}_1(\cdot) \equiv \mathcal{G}(\cdot)$	154
$\widetilde{\mathcal{G}}_j(\cdot)$	squared gain function for $\{\tilde{g}_{j,l}\}$, with $\widetilde{\mathcal{G}}_1(\cdot) \equiv \widetilde{\mathcal{G}}(\cdot)$	202
$\mathcal{G}^{(\mathrm{D})}(\cdot)$	squared gain function for Daubechies scaling filter $\{g_l\}$	105
$\{g_l\}$	DWT scaling filter	75, 154, 463
$\{\tilde{g}_l\}$	MODWT scaling filter	163, 202
$\{g_l^\circ\}$	$\{g_l\}$ periodized to length N	77
$\{\tilde{g}_l^\circ\}$	$\{\tilde{g}_l\}$ periodized to length N	168
$\{\bar{g}_l\}$	reversed scaling filter, i.e., $\bar{g}_l = g_{-l}$	463
$\{g_l^{(\mathrm{ep})}\}$	extremal phase (minimum delay) Daubechies scaling filter	106
$\{g_l^{(\mathrm{la})}\}$	least asymmetric (LA) Daubechies scaling filter	107
$\{g_{j,l}\}$	jth level DWT scaling filter, with $\{g_{1,l}\} \equiv \{g_j\}$	96, 154
$\{\tilde{g}_{j,l}\}$	jth level MODWT scaling filter, with $\{\tilde{g}_{1,l}\} \equiv \{\tilde{g}_j\}$	169, 202
$\{g_{j,l}^\circ\}$	$\{g_{j,l}\}$ periodized to length N	97
$\{\tilde{g}_{j,l}^\circ\}$	$\{\tilde{g}_{j,l}\}$ periodized to length N	170
H	Hurst coefficient	279, 286
$H(\cdot)$	transfer function for $\{h_l\}$	69, 154
$\widetilde{H}(\cdot)$	transfer function for $\{\tilde{h}_l\}$	163, 202
$H_j(\cdot)$	transfer function for $\{h_{j,l}\}$, with $H_1(\cdot) \equiv H(\cdot)$	96, 154
$\widetilde{H}_j(\cdot)$	transfer function for $\{\tilde{h}_{j,l}\}$, with $\widetilde{H}_1(\cdot) \equiv \widetilde{H}(\cdot)$	169, 202
$\mathcal{H}(\cdot)$	squared gain function for $\{h_l\}$	69, 154
$\widetilde{\mathcal{H}}(\cdot)$	squared gain function for $\{\tilde{h}_l\}$	163, 202
$\mathcal{H}_j(\cdot)$	squared gain function for $\{h_{j,l}\}$, with $\mathcal{H}_1(\cdot) \equiv \mathcal{H}(\cdot)$	154
$\widetilde{\mathcal{H}}_j(\cdot)$	squared gain function for $\{\tilde{h}_{j,l}\}$, with $\widetilde{\mathcal{H}}_1(\cdot) \equiv \widetilde{\mathcal{H}}(\cdot)$	202
$\mathcal{H}^{(\mathrm{D})}(\cdot)$	squared gain function for Daubechies wavelet filter $\{h_l\}$	105
$\{h_l\}$	DWT wavelet filter	68–9, 154, 474
$\{\tilde{h}_l\}$	MODWT wavelet filter	163, 202
$\{h_l^\circ\}$	$\{h_l\}$ periodized to length N	70–1
$\{\tilde{h}_l^\circ\}$	$\{\tilde{h}_l\}$ periodized to length N	167–8
$\{\bar{h}_l\}$	reversed wavelet filter, i.e., $\bar{h}_l = h_{-l}$	472, 474
$\{h_{j,l}\}$	jth level DWT wavelet filter, with $\{h_{1,l}\} \equiv \{h_j\}$	95, 154
$\{\tilde{h}_{j,l}\}$	jth level MODWT wavelet filter, with $\{\tilde{h}_{1,l}\} \equiv \{\tilde{h}_j\}$	169, 202
$\{h_{j,l}^\circ\}$	$\{h_{j,l}\}$ periodized to length N	96
$\{\tilde{h}_{j,l}^\circ\}$	$\{\tilde{h}_{j,l}\}$ periodized to length N	170
I_N	$N \times N$ identity matrix	42
$\Im(z)$	imaginary part of complex-valued number z	21
i	$\sqrt{-1}$	20

J	largest DWT level for sample size $N = 2^J$	57
J_0	level of partial DWT or of MODWT	104, 145, 169, 199
j	level (index) for scale usually (also used as generic index)	59
k	index for frequency usually (also used as generic index)	46
L	width of wavelet or scaling filter (unit scale)	68
L_j	width of jth level equivalent wavelet or scaling filter	96
L'_j	number of jth level DWT boundary coefficients	146
$L^2(\mathbb{R})$	set of square integrable real-valued functions	458
$\log_{10}(\cdot)$, $\log(\cdot)$	log base 10, log base e	73, 400–1
M_j	number of nonboundary jth level MODWT coefficients	306
$M(\mathbf{W}_{j,n})$	cost of DWPT vector $\mathbf{W}_{j,n}$	223
$m(\cdot)$	additive cost functional	223
$m \bmod N$	m modulo N	30
$m + n \bmod N$	$(m + n)$ modulo N	30
N	sample size	28, 41
N_j	$N/2^j$, number of jth level DWT coefficients	94
$\mathcal{N}(\mu, \sigma^2)$	Gaussian (normal) RV with mean μ and variance σ^2	257
n_l	lth component of vector $\mathbf{n}$	403
$\mathbf{n}$	transform coefficients for non-IID noise $\boldsymbol{\eta}$	403
O_l	lth element of $\mathbf{O}$	43, 398
$O_l^{(\mathrm{ht})}, O_l^{(\mathrm{st})}, O_l^{(\mathrm{mt})}$	result of applying hard/soft/mid thresholding to O_l	399–400
$\mathcal{O}$	$N \times N$ orthonormal transform matrix	42
$\mathbf{O}$	transform coefficients obtained using $\mathcal{O}$	43
$P_{\mathcal{F}}(f_k)$	discrete Fourier empirical power spectrum	48
$P_{\mathcal{W}}(\tau_j)$	discrete wavelet empirical power spectrum (DWT)	62
$P_{\widetilde{\mathcal{W}}}(\tau_j)$	discrete wavelet empirical power spectrum (MODWT)	180
$\mathcal{P}_j$	transform matrix for jth stage of DWT pyramid algorithm	94
$\widetilde{\mathcal{P}}_j$	like $\mathcal{P}_j$, but for MODWT pyramid algorithm	176
$\mathbf{P}[A]$	probability that the event A will occur	256
$Q_\eta(p)$	$p \times 100\%$ percentage point of χ^2_η distribution	263–4
$R_{j,t}$	tth component on jth level of $\mathbf{R}$ for DWT	424
R_l	lth component of vector $\mathbf{R}$	407
$\mathcal{R}_j$	jth level wavelet rough (DWT)	66
$\mathbb{R}$	the entire real axis	457
$\mathbb{R}^N$	space of real-valued N dimensional vectors	45
$\Re(z)$	real part of complex-valued number z	21
$\mathbf{R}$	transform coefficients for stochastic signal $\mathbf{C}$	407
$S_j(\cdot)$	SDF for $\{\overline{W}_{j,t}\}$ or for nonboundary part of $\{W_{j,t}\}$	304, 348
$S_X(\cdot)$	(power) spectral density function (SDF)	267
$\hat{S}_X^{(\mathrm{mt})}(\cdot)$	multitaper SDF estimator	274
$\hat{S}_{X,n}^{(\mathrm{mt})}(\cdot)$	nth eigenspectrum used to form $\hat{S}_X^{(\mathrm{mt})}(\cdot)$	274

$\hat{S}_X^{(\mathrm{p})}(\cdot)$	periodogram	269
$\mathcal{S}_J$	Jth level wavelet smooth (DWT)	64
$\widetilde{\mathcal{S}}_{J_0}$	J_0th level wavelet smooth (MODWT)	169, 171
$\{s_{X,\tau}\}$	autocovariance sequence (ACVS)	266
$\{\hat{s}_{X,\tau}^{(\mathrm{p})}\}$	'biased' estimator of ACVS	269
$\mathcal{T}$	$N \times N$ circular shift matrix or unit delay operator	52, 457
t	actual time (continuous) or a unitless index (discrete)	5, 24
$U_{j,n}(\cdot)$	transfer function for $\{u_{j,n,l}\}$	215
$\widetilde{U}_{j,n}(\cdot)$	transfer function for $\{\tilde{u}_{j,n,l}\}$	232
$\{u_{j,n,l}\}$	DWPT filter for node (j,n)	214
$\{\tilde{u}_{j,n,l}\}$	MODWPT filter for node (j,n)	231
V_j	approximation subspace for functions of scale λ_j	462
$V_{j,t}$	tth element of $\mathbf{V}_j$	94
$\widetilde{V}_{j,t}$	tth element of $\widetilde{\mathbf{V}}_j$	169
$\mathcal{V}_j$	$N_j \times N$ matrix mapping $\mathbf{X}$ to $\mathbf{V}_j$	94
$\widetilde{\mathcal{V}}_j$	$N \times N$ matrix mapping $\mathbf{X}$ to $\widetilde{\mathbf{V}}_j$	171
$\mathbf{V}_j$	vector of jth level DWT scaling coefficients	94
$\widetilde{\mathbf{V}}_j$	vector of jth level MODWT scaling coefficients	169
$\mathrm{var}\,\{\cdot\}$	variance operator	259
W_j	detail subspace for functions of scale τ_j	472
$W_{j,n,t}$	tth element of $\mathbf{W}_{j,n}$	214
$\widetilde{W}_{j,n,t}$	tth element of $\widetilde{\mathbf{W}}_{j,n}$	231
$W_{j,t}$	tth element of $\mathbf{W}_j$	94
$\widetilde{W}_{j,t}$	tth element of $\widetilde{\mathbf{W}}_j$	169
$\{\overline{W}_{j,t}\}$	jth level MODWT coefficients for stochastic process $\{X_t\}$	296
W_n	nth DWT coefficient (nth element in $\mathbf{W}$)	57
$\mathcal{W}$	$N \times N$ discrete wavelet transform matrix	57
$\mathcal{W}_j$	$N_j \times N$ matrix mapping $\mathbf{X}$ to $\mathbf{W}_j$ (submatrix of $\mathcal{W}$)	94
$\widetilde{\mathcal{W}}_j$	$N \times N$ matrix mapping $\mathbf{X}$ to $\widetilde{\mathbf{W}}_j$	171
$\mathbf{W}$	vector containing DWT coefficients $\{W_n\}$	57, 150
$\mathbf{W}_j$	vector of jth level DWT wavelet coefficients (part of $\mathbf{W}$)	94
$\widetilde{\mathbf{W}}_j$	vector of jth level MODWT wavelet coefficients	169
$\mathbf{W}_{j,n}$	vector of DWPT coefficients at node (j,n)	209
$\widetilde{\mathbf{W}}_{j,n}$	vector of MODWPT coefficients at node (j,n)	232
$\mathrm{width_a}\{\cdot\}$	autocorrelation width	12, 103
$X_0, \ldots, X_{N-1}$	time series or portion of a stochastic process	41, 269
$\{X_t\}$	time series or stochastic process	41, 266, 295–6
$\overline{X}$	sample mean (arithmetic average) of $X_0, \ldots, X_{N-1}$	48
$\overline{X}_t(\lambda)$	sample mean of $X_{t-\lambda+1}, X_{t-\lambda+2}, \ldots, X_t$	58
$\{\mathcal{X}_k\}$	discrete Fourier transform of $\{X_t\}$	72
$\mathbf{X}$	vector containing $X_0, \ldots, X_{N-1}$	41–2

$Y^{(\mathrm{mt})}(f)$	log multitaper SDF estimate plus a constant	276
$Y^{(\mathrm{p})}(f)$	log periodogram plus a constant	271
Z	Gaussian (normal) RV with unit mean and zero variance	257

- *Greek notation used frequently*

α	exponent of power law spectral density function	279, 281, 286
α	significance level of a test	373, 434
β	slope in linear regression model related to FD parameter δ	374
$\hat{\beta}^{(\mathrm{wls})}$	WLSE of β	376
$\Gamma(\cdot)$	gamma function	257
$\boldsymbol{\gamma}$	index for vectors in matching pursuit dictionary	239
γ	Euler's constant $(0.577215664901532\cdots)$	270, 432
$\gamma_{J_0}^{(G)}$, $\gamma_j^{(H)}$	index of coefficient earliest in time in $\mathbf{V}_{J_0}$, $\mathbf{W}_j$	137, 147
$\bar{\gamma}_{J_0}^{(G)}$, $\bar{\gamma}_j^{(H)}$	number of 'early' boundary coefficients in $\mathbf{V}_{J_0}$, $\mathbf{W}_j$	137, 147
γ_l^2	ratio of component variances in Gaussian mixture model	410
$\gamma(\cdot)$	real-valued function	457
$\gamma_{j,k}(\cdot)$	translated and dilated version of $\gamma(\cdot)$	459
Δt	sampling interval	48, 59
δ	generic threshold	223, 399
δ	long memory parameter for FD process	283–4, 286, 288
$\delta^{(\mathrm{s})}$	long memory parameter for stationary FD process	288, 368
$\delta^{(\mathrm{S})}$	threshold based on Stein's unbiased risk estimator	405
$\delta^{(\mathrm{u})}$	universal threshold	400
$\delta_{j,k}$	Kronecker delta function	42–3
$\hat{\delta}$	exact MLE of δ for stationary FD process	368
$\hat{\delta}^{(\mathrm{loocv})}$	threshold for leave-one-out cross-validation	402, 423
$\hat{\delta}^{(\mathrm{tfcv})}$	threshold for two-fold cross-validation	402, 422
$\hat{\delta}^{(\mathrm{wls})}$	WLSE of δ for stationary or nonstationary FD process	377
$\tilde{\delta}^{(\mathrm{s})}$	approximate MLE of δ for stationary FD process	363
$\tilde{\delta}^{(\mathrm{s/ns})}$	like $\tilde{\delta}^{(\mathrm{s})}$, but for nonstationary FD processes also	371
$\boldsymbol{\epsilon}$	N dimensional vector containing IID RVs	393
ϵ	a small positive number	2, 486
$\epsilon(f)$, $\epsilon(f_k)$	error term in frequency domain model (uncorrelated)	270, 432
ε_t	tth term in sequence of uncorrelated RVs (white noise)	268
ζ	intercept in linear regression model	374
$\boldsymbol{\eta}$	N dimensional vector containing non-IID RVs	393
η	degrees of freedom in a chi-square distribution	263, 313
$\eta_1, \eta_2, \eta_3, \eta_j$	EDOFs for wavelet variance estimator	313–4, 376
$\eta(f)$, $\eta(f_k)$	error term in frequency domain model (correlated)	276, 440
θ	argument in polar representation $z = \lvert z\rvert e^{i\theta}$	21
θ	parameter with prior distribution in Bayesian model	264

$\theta(\cdot)$	phase function for a filter	25
$\theta^{(G)}(\cdot)$	phase function for DWT scaling filter	106
$\theta^{(H)}(\cdot)$	phase function for DWT wavelet filter	112
$\theta_{c_{j,n,m}}(\cdot)$	component of phase function for DWPT wavelet filter	229
ϑ	degrees of freedom in a t distribution	257, 426
κ	scale parameter in a generalized t distribution	258, 414
κ	RV uniformly distributed over integers $0, 1, \ldots, N-1$	356
Λ_N	$N \times N$ diagonal covariance matrix	355
λ	scale (length of an interval or of an average)	6, 58
λ_j	2^j, unitless scale of jth level scaling coefficients ($j \geq 1$)	85, 481
μ	expected value of a random variable	256–7
ν	lag in frequency domain autocovariance	276–7
ν	advance for time series or filter	111–2
$\nu_j^{(G)}, \nu_j^{(H)}$	advance for scaling filter, wavelet filter	114
$\nu_{j,n}$	advance for wavelet packet filter	229
$\nu_X^2(\tau_j)$	wavelet variance at scale τ_j (time independent)	296
$\hat{\nu}_X^2(\tau_j)$	unbiased MODWT estimator of wavelet variance at scale τ_j	306
$\hat{\hat{\nu}}_X^2(\tau_j)$	unbiased DWT estimator of wavelet variance at scale τ_j	308
$\tilde{\nu}_X^2(\tau_j)$	biased MODWT estimator of wavelet variance at scale τ_j	306
$\tilde{\tilde{\nu}}_X^2(\tau_j)$	biased DWT estimator of wavelet variance at scale τ_j	308
π	$3.141592653589793\cdots$	3, 21
ρ	correlation between two RVs	259
$\rho_{X,\tau}$	autocorrelation sequence for stationary process at lag τ	266
$\hat{\rho}_{X,\tau}$	estimator of autocorrelation sequence at lag τ	16, 341
$\Sigma_{\mathbf{X}}$	covariance matrix for vector $\mathbf{X}$ of RVs	259, 262
$\widetilde{\Sigma}_{\mathbf{X}}$	wavelet-based approximation to covariance matrix $\Sigma_{\mathbf{X}}$	362
σ^2, σ_X^2	variance of a random variable	3, 257, 279
σ_ϵ^2	variance of an IID process	393
σ_ε^2	variance of a white noise process	268
$\sigma_{G_l}^2$	variance of one part of a Gaussian mixture model	410
$\sigma_{n_l}^2$	variance of a non-IID process	403
$\hat{\sigma}_X^2, \hat{\sigma}_Y^2$	sample variance formed using sample mean	48, 299
$\tilde{\sigma}_Y^2$	sample variance formed using process mean	299
$\hat{\sigma}_{(\mathrm{mad})}^2$	estimator of variance formed using MAD	420
$\tilde{\sigma}_{(\mathrm{mad})}^2$	like $\hat{\sigma}_{(\mathrm{mad})}^2$, but based on MODWT	429
τ	lag index in autocorrelation or autocovariance sequence	16, 266
τ_j	2^{j-1}, unitless scale of jth level wavelet coefficients ($j \geq 1$)	59
$\Upsilon(\cdot)$	function whose minimization yields SURE threshold $\delta^{(\mathrm{S})}$	405
υ, υ_l	inverse variance of Laplace distribution	257, 265, 413
$\phi(\cdot)$	scaling function	459
$\phi_{j,k}(\cdot)$	translated and dilated scaling function	460

$\phi^{(\mathrm{H})}(\cdot)$	Haar scaling function 460
$\phi_{p,1}, \ldots, \phi_{p,p}$	coefficients of an AR(p) process 268, 292
χ^2_η	chi-square random variable with η degrees of freedom 263
$\psi(\cdot)$	wavelet function 2, 474
$\psi(\cdot)$, $\psi'(\cdot)$	digamma function, trigamma function 275, 376, 440
$\psi_{j,k}(\cdot)$	translated and dilated wavelet function 474
$\psi^{(\mathrm{H})}(\cdot)$	Haar wavelet function 2, 475
$\psi^{(\mathrm{Mh})}(\cdot)$	Mexican hat wavelet function 3
ω_0	parameter for Morlet wavelet function 4

- *Other mathematical conventions and symbols used frequently*

$\approx$	approximately equal to 83–4, 264, 297
$\{a \star a_t\}$	autocorrelation of real-valued sequence $\{a_t\}$ 69
$\{a^* \star a_t\}$	autocorrelation of complex-valued sequence $\{a_t\}$ 25, 30, 36–7
$\mathcal{O}^H$	complex conjugate (Hermitian) transpose of matrix $\mathcal{O}$ 45
z^*	complex conjugate of z 21
$\{a^* \star b_t\}$	complex cross-correlation of sequences $\{a_t\}$ and $\{b_t\}$ 24–5, 30
$\in$, $\notin$	contained in, not contained in 2, 398
$\{a * b_t\}$	convolution of sequences $\{a_t\}$ and $\{b_t\}$ 24, 30, 36–7
$\lvert\Sigma_{\mathbf{X}}\rvert$	determinant of matrix $\Sigma_{\mathbf{X}}$ 361
$\downarrow$	downsampling (removing values from a sequence) ... 70, 80, 92, 96
$\doteq$	equal at the stated precision (e.g., $\pi \doteq 3.14$ or $\pi \doteq 3.1416$) ... 3, 73
$\equiv$	equal by definition 20
$\stackrel{\mathrm{d}}{=}$	equal in distribution 257
$\hat{}$	estimator or estimate; e.g., $\hat{\nu}^2_X(\tau_j)$ is an estimator of $\nu^2_X(\tau_j)$ ... 306
$\{a_t\} \longleftrightarrow \{A_k\}$	Fourier transform pair ($\{a_t\}$ is a finite sequence) 29, 36
$\{a_t\} \longleftrightarrow A(\cdot)$	Fourier transform pair ($\{a_t\}$ is an infinite sequence) 23, 35
$\lfloor x \rfloor$, $\lceil x \rceil$	greatest integer $\leq x$, smallest integer $\geq x$ 50, 146
$1_{\mathcal{J}}(\cdot)$	indicator function for set $\mathcal{J}$ 404
$\langle \cdot, \cdot \rangle$	inner product 42, 45
$\lvert z \rvert$	modulus (absolute value or magnitude) of z 21
$\mathbf{1}$	N dimensional vector of ones 50
$\{a_t^\circ\}$	periodized version (to length N) of infinite sequence $\{a_t\}$ 33
$[a, b]$	set of values x such that $a \leq x \leq b$ 22
(a, b)	set of values x such that $a < x < b$ 2
$(a, b]$	set of values x such that $a < x \leq b$ 465
$\lVert \cdot \rVert^2$	squared norm 42, 46
$\mathbf{X}^T$, $\mathcal{O}^T$	transpose of vector $\mathbf{X}$, transpose of matrix $\mathcal{O}$ 42
$\uparrow$	upsampling (inserting zeros into a sequence) 82, 95, 201
$\mathcal{O}_{j\bullet}$	vector containing elements of jth row of $N \times N$ matrix $\mathcal{O}$ 42
$\mathcal{O}_{\bullet k}$	vector containing elements of kth column of $N \times N$ matrix $\mathcal{O}$ .. 42
$\mathbf{0}$, $\mathbf{0}_j$	vector of zeros 101

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