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BOND MATH

THE THEORY BEHIND THE FORMULAS

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To my students

Preface

This book could be titled *Applied Bond Math* or, perhaps, *Practical Bond Math*. Those who do serious research on fixed-income securities and markets know that this subject matter goes far beyond the mathematics covered herein. Those who are interested in discussions about "pricing kernels" and "stochastic discount rates" will have to look elsewhere. My target audience is those who work in the finance industry (or aspire to), know what a Bloomberg page is, and in the course of the day might hear or use terms such as "yield to maturity," "forward curve," and "modified duration."

My objective in *Bond Math* is to explain the theory and assumptions that lie behind the commonly used statistics regarding the risk and return on bonds. I show many of the formulas that are used to calculate yield and duration statistics and, in the Technical Appendix, their formal derivations. But I do not expect a reader to actually *use* the formulas or *do* the calculations. There is much to be gained by recognizing that "there exists an equation" and becoming more comfortable using a number that is taken from a Bloomberg page, knowing that the result could have been obtained using a bond math formula.

This book is based on my 25 years of experience teaching this material to graduate students and finance professionals. For that, I thank the many deans, department chairs, and program directors at the Boston University School of Management who have allowed me to continue teaching fixed-income courses over the years. I thank Euromoney Training in New York and Hong Kong for organizing four-day intensive courses for me all over the world. I thank training coordinators at Chase Manhattan Bank (and its heritage banks, Manufacturers Hanover and Chemical), Lehman Brothers, and the Bank of Boston for paying me handsomely to teach their employees on so many occasions in so many interesting venues. Bond math has been very, very good to me.

The title of this book emanates from an eponymous two-day course I taught many years ago at the old Manny Hanny. (Okay, I admit that I have always wanted to use the word "eponymous"; now I can cross that off of my bucket list.) I thank Keith Brown of the University of Texas at Austin, who co-designed and co-taught many of those executive training courses, for emphasizing the value of relating the formulas to results reported on Bloomberg. I have found that users of "black box" technologies find comfort in knowing how those bond numbers are calculated, which ones are useful, which ones are essentially meaningless, and which ones are just wrong.

Our journey through applied and practical bond math starts in the money market, where we have to deal with anachronisms like discount rates and a 360-day year. A key point in Chapter 1 is that knowing the *periodicity* of an annual interest rate (i.e., the assumed number of periods in the year) is critical. Converting from one periodicity to another—for instance, from quarterly to semiannual—is a core bond math calculation that I use throughout the book. Money market rates can be deceiving because they are not intuitive and do not follow classic time-value-of-money principles taught in introductory finance courses. You have to know what you are doing to play with T-bills, commercial paper, and bankers acceptances.

Chapters 2 and 3 go deep into calculating prices and yields, first on zero-coupon bonds to get the ideas out for a simple security like U.S. Treasury STRIPS (i.e., just two cash flows) and then on coupon bonds for which coupon reinvestment is an issue. The yield to maturity on a bond is a *summary statistic* about its cash flows—it's important to know the assumptions that underlie this widely quoted measure of an investor's rate of return and what to do when those assumptions are untenable. I decipher Bloomberg's Yield Analysis page for a typical corporate bond, showing the math behind "street convention," "U.S. government equivalent," and "true" yields. The problem is distinguishing between yields that are pure data (and can be overlooked) and those that provide information useful in making a decision about the bond.

Chapter 4 continues the exploration of rate-of-return measures on an after-tax basis for corporate, Treasury, and municipal bonds. Like all tax matters, this necessarily gets technical and complicated. Taxation, at least in the U.S., depends on when the bond was issued (there were significant changes in the 1980s and 1990s), at what issuance price (there are different rules for original issue discount bonds), and whether a bond issued at (or close to) par value is later purchased at a premium or discount. Given the inevitability of taxes, this is important stuff—and it is stuff on which Bloomberg sometimes reports a misleading result, at least for U.S. investors.

Yield curve analysis, in Chapter 5, is arguably the most important topic in the book. There are many practical applications arising from bootstrapped implied zero-coupon (or spot) rates and implied forward rates—identifying arbitrage opportunities, obtaining discount factors to get present values, calculating spreads, and pricing and valuing derivatives. However, the operative assumption in this analysis is "no arbitrage"—that is, transactions costs and counterparty credit risk are sufficiently small so that trading eliminates any arbitrage opportunity. Therefore, while mathematically elegant, yield curve analysis is best applied to Treasury securities and LIBOR-based interest rate derivatives for which the no-arbitrage assumption is reasonable.

Duration and convexity, the subject of Chapter 6, is the most mathematical topic in this book. These statistics, which in classic form measure the sensitivity of the bond price to a change in its yield to maturity, can be derived with algebra and calculus. Those details are relegated to the Technical Appendix. Another version of the risk statistics measures the sensitivity of the bond price to a shift in the entire Treasury yield curve. I call the former *yield* and the latter *curve* duration and convexity and demonstrate where and how they are presented on Bloomberg pages.

Chapters 7 and 8 examine floating-rate notes (floaters), inflation-indexed bonds (linkers), and interest rate swaps. The idea is to use the bond math toolkit—periodicity conversions, bond valuation, after-tax rates of return, implied spot rates, implied forward rates, and duration and convexity—to examine securities other than traditional fixed-rate and zero-coupon bonds. In particular, I look for circumstances of *negative duration*, meaning market value and interest rates are positively correlated. That's an obvious feature for one type of interest rate swap but a real oddity for a floater and a linker.

Understanding the risk and return characteristics for an individual bond is easy compared to a portfolio of bonds. In Chapter 9, I show different ways of getting summary statistics. One is to treat the portfolio as a big bundle of cash flow and derive its yield, duration, and convexity is if it were just a single bond with many variable payments. While that is theoretically correct, in practice portfolio statistics are calculated as weighted averages of those for the constituent bonds. Some statistics can be aggregated in this manner and provide reasonable estimates of the "true" values, depending on how the weights are calculated and on the shape of the yield curve.

Chapter 10 is on bond strategies. If your hope is that I'll show you how to get rich by trading bonds, you'll be disappointed. My focus is on how the bond math tools and the various risk and return statistics that we can calculate for individual bonds and portfolios can facilitate either aggressive or passive investment strategies. I'll discuss derivative overlays, immunization, and liability-driven investing and conclude with a request that the finance industry create target-duration bond funds.

I'd like to thank my Wiley editors for allowing me to deviate from their usual publishing standards so that I can use in this book acronyms, italics, and notation as I prefer. Now let's get started in the money market.

Contents

CHAPTER 1

Money Market Interest Rates

An interest rate is a summary statistic about the cash flows on a debt security such as a loan or a bond. As a statistic, it is a number that we calculate. An objective of this chapter is to demonstrate that there are many ways to do this calculation. Like many statistics, an interest rate can be deceiving and misleading. Nevertheless, we need interest rates to make financial decisions about borrowing and lending money and about buying and selling securities. To avoid being deceived or misled, we need to understand how interest rates are calculated.

It is useful to divide the world of debt securities into *short-term money markets* and *long-term bond markets*. The former is the home of money market instruments such as Treasury bills, commercial paper, bankers acceptances, bank certificates of deposit, and overnight and term sale-repurchase agreements (called "repos"). The latter is where we find coupon-bearing notes and bonds that are issued by the Treasury, corporations, federal agencies, and municipalities. The key reference interest rate in the U.S. money market is 3-month LIBOR (the London inter-bank offered rate); the benchmark bond yield is on 10-year Treasuries.

This chapter is on money market interest rates. Although the money market usually is defined as securities maturing in one year or less, much of the activity is in short-term instruments, from overnight out to six months. The typical motivation for both issuers and investors is cash management arising from the mismatch in the timing of revenues and expenses. Therefore, primary investor concerns are liquidity and safety. The instruments themselves are straightforward and entail just two cash flows, the purchase price and a known redemption amount at maturity.

Let's start with a practical money market investment problem. A fund manager has about $1 million to invest and needs to choose between two 6-month securities: (1) commercial paper (CP) quoted at 3.80% and (2) a bank certificate of deposit (CD) quoted at 3.90%. Assuming that the credit risks are the same and any differences in liquidity and taxation are immaterial, which investment offers the better rate of return, the CP at 3.80% or the CD at 3.90%? To the uninitiated, this must seem like a trick question—surely, 3.90% is higher than 3.80%. If we are correct in our assessment that the risks are the same, the CD appears to pick up an extra 10 basis points. The initiated know that first it is time for a bit of bond math.

Interest Rates in Textbook Theory

You probably were first introduced to the time value of money in college or in a job training program using equations such as these:

$$FV = PV * (1+i)^N \text{ and } PV = \frac{FV}{(1+i)^N} \tag{1.1}$$

where FV = future value, PV = present value, i = interest rate per time period, and N = number of time periods to maturity.

The two equations are the same, of course, and merely are rearranged algebraically. The future value is the present value moved forward along a time trajectory representing compound interest over the N periods; the present value is the future value discounted back to day zero at rate i per period.

In your studies, you no doubt worked through many time-value-of-money problems, such as: How much will you accumulate after 20 years if you invest $1,000 today at an annual interest rate of 5%? How much do you need to invest today to accumulate $10,000 in 30 years assuming a rate of 6%? You likely used the time-value-of-money keys on a financial calculator, but you just as easily could have plugged the numbers into the equations in 1.1 and solved via the arithmetic functions.

$$\$1{,}000 * (1.05)^{20} = \$2{,}653 \text{ and } \frac{\$10{,}000}{(1.06)^{30}} = \$1{,}741$$

The interest rate in standard textbook theory is well defined. It is the *growth rate of money* over time—it describes the trajectory that allows $1,000 to grow to \$2,653 over 20 years. You can interpret an interest rate as an *exchange rate across time*. Usually we think of an exchange rate as a trade between two currencies (e.g., a spot or a forward foreign exchange rate between the U.S. dollar and the euro). An interest rate tells you the amounts in the same currency that you would accept at different points in time. You would be indifferent between \$1,741 now and \$10,000 in 30 years, assuming that 6% is the correct exchange rate for you. An interest rate also indicates the *price of money*. If you want or need \$1,000 today, you have to pay 5% annually to get it, assuming you will make repayment in 20 years.

Despite the purity of an interest rate in time-value-of-money analysis, you cannot use the equations in 1.1 to do interest rate and cash flow calculations on money market securities. This is important: *Money market interest rate calculations do not use textbook time-value-of-money equations*. For a money manager who has \$1,000,000 to invest in a bank CD paying 3.90% for half of a year, it is *wrong* to calculate the future value in this manner:

$$\$1{,}000{,}000 * (1.0390)^{0.5} = \$1{,}019{,}313$$

While it is tempting to use $N = 0.5$ in equation 1.1 for a 6-month CD, it is not the way money market instruments work in the real world.

Money Market Add-on Rates

There are two distinct ways that money market rates are quoted: as an *add-on rate* and as a *discount rate*. Add-on rates generally are used on commercial bank loans and deposits, including certificates of deposit, repos, and fed funds transactions. Importantly, LIBOR is quoted on an add-on rate basis. Discount rates in the U.S. are used with T-bills, commercial paper, and bankers acceptances. However, there are no hard-and-fast rules regarding rate quotation in domestic or international markets. For example, when commercial paper is issued in the Euromarkets, rates typically are on an add-on basis, not a discount rate basis. The Federal Reserve lends money to commercial banks at its official "discount rate." That interest rate, however, actually is quoted as an add-on rate, not as a discount rate. Money market rates can be confusing—when in doubt, verify!

First, let's consider rate quotation on a bank certificate of deposit. Add-on rates are logical and follow *simple interest* calculations. The interest is added on to the principal amount to get the redemption payment at maturity. Let *AOR* stand for add-on rate, *PV* the present value (the initial principal amount), *FV* the future value (the redemption payment including interest), *Days* the number of days until maturity, and *Year* the number of days in the year. The relationship between these variables is:

$$FV = PV + \left[PV * AOR * \frac{Days}{Year} \right] \tag{1.2}$$

The term in brackets is the interest earned on the bank CD—it is just the initial principal times the annual add-on rate times the fraction of the year.

The expression in 1.2 can be written more succinctly as:

$$FV = PV * \left[1 + \left(AOR * \frac{Days}{Year} \right) \right] \tag{1.3}$$

Now we can calculate accurately the future value, or the redemption amount including interest, on the $1,000,000 bank CD paying 3.90% for six months. But first we have to deal with the fraction of the year. Most money market instruments in the U.S. use an "actual/360" day-count convention. That means *Days*, the numerator, is the actual number of days between the settlement date when the CD is purchased and the date it matures. The denominator usually is 360 days in the U.S. but in many other countries a more realistic 365-day year is used. Assuming that *Days* is 180 and *Year* is 360, the future value of the *CD* is $1,019,500, and not $1,019,313 as incorrectly calculated using the standard time-value-of-money formulation.

$$FV = \$1{,}000{,}000 * \left[1 + \left(0.0390 * \frac{180}{360} \right) \right] = \$1{,}019{,}500$$

Once the bank CD is issued, the *FV* is a known, fixed amount. Suppose that two months go by and the investor—for example, a money market mutual fund—decides to sell. A securities dealer at that time quotes a bid rate of 3.72% and an asked (or offered) rate of 3.70% on 4-month CDs corresponding to the credit risk of the issuing bank. Note that securities in the money market trade on a *rate basis*. The bid rate is higher than the ask rate so that the security will be bought by the dealer at a lower price than it is sold. In the bond market, securities usually trade on a *price basis*.

The sale price of the CD after the two months have gone by is found by substituting $FV = \$1{,}019{,}500$, $AOR = 0.0372$, and $Days = 120$ into equation 1.3.

$$\$1{,}019{,}500 = PV * \left[1 + \left(0.0372 * \frac{120}{360}\right)\right], \quad PV = \$1{,}007{,}013$$

Note that the dealer buys the CD from the mutual fund at its quoted bid rate. We assume here that there are actually 120 days between the settlement date for the transaction and the maturity date. In most markets, there is a one-day difference between the trade date and the settlement date (i.e., next-day settlement, or "T + 1").

The general pricing equation for add-on rate instruments shown in 1.3 can be rearranged algebraically to isolate the *AOR* term.

$$AOR = \left(\frac{Year}{Days}\right) * \left(\frac{FV - PV}{PV}\right) \tag{1.4}$$

This indicates that a money market add-on rate is an annual percentage rate (APR) in that it is the number of time periods in the year, the first term in parentheses, times the interest rate per period, the second term. $FV - PV$ is the interest earned; that divided by amount invested PV is the rate of return on the transaction for that time period. To annualize the periodic rate of return, we simply multiply by the number of periods in the year (*Year*/*Days*). I call this the *periodicity* of the interest rate. If *Year* is assumed to be 360 days and *Days* is 90, the periodicity is 4; if *Days* is 180, the periodicity is 2. Knowing the periodicity is critical to understanding an interest rate.

APRs are widely used in both money markets and bond markets. For example, the typical fixed-income bond makes semiannual coupon payments. If the payment is \$3 per \$100 in par value on May 15th and November 15th of each year, the coupon rate is stated to be 6%. Using an APR in the money market does require a subtle yet important assumption, however. It is assumed implicitly that the transaction can be replicated at the same rate per period. The 6-month bank CD in the example can have its *AOR* written like this:

$$AOR = \left(\frac{360}{180}\right) * \left(\frac{\$1{,}019{,}500 - \$1{,}000{,}000}{\$1{,}000{,}000}\right) = 0.0390$$

The periodicity on this CD is 2 and its rate per (6-month) time period is 1.95%. The annualized rate of 3.90% assumes replication of the 6-month transaction on the very same terms.

Equation 1.4 can be used to obtain the ex-post rate of return realized by the money market mutual fund that purchased the CD and then sold it two months later to the dealer. Substitute in $PV = \$1{,}000{,}000$, $FV = \$1{,}007{,}013$, and $Days = 60$.

$$AOR = \left(\frac{360}{60}\right) * \left(\frac{\$1{,}007{,}013 - \$1{,}000{,}000}{\$1{,}000{,}000}\right) = 0.0421$$

The 2-month holding-period rate of return turns out to be 4.21%. Notice that in this series of calculations, the meanings of *PV* and *FV* change. In one case *PV* is the original principal on the CD, in another it is the market value at a later date. In one case *FV* is the redemption amount at maturity, in another it is the sale price prior to maturity. Nevertheless, *PV* is always the first cash flow and *FV* is the second.

The mutual fund buys a 6-month CD at 3.90%, sells it as a 4-month CD at 3.72%, and realizes a 2-month holding-period rate of return of 4.21%. This statement, while accurate, contains rates that are annualized for different periodicities. Here 3.90% has a periodicity of 2, 3.72% has a periodicity of 3, and 4.21% has a periodicity of 6. Comparing interest rates that have varying periodicities can be a problem but one that can be remedied with a conversion formula. But first we need to deal with another problem—money market discount rates.

Money Market Discount Rates

Treasury bills, commercial paper, and bankers acceptances in the U.S. are quoted on a discount rate (*DR*) basis. The price of the security is a discount from the face value.

$$PV = FV - \left[FV * DR * \frac{Days}{Year} \right] \tag{1.5}$$

Here, *PV* and *FV* are the two cash flows on the security; *PV* is the current price and *FV* is the amount paid at maturity. The term in brackets is the amount of the discount—it is the future (or face) value times the annual